# UNIQUE SOLUTIONS TO POWER-TRANSFORMED AFFINE SYSTEMS

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ABSTRACT. Systems of the form  $x = (Ax^s)^{1/s} + b$  arise in a range of economic and financial applications, where A is a linear operator acting on a space of real-valued functions (or vectors) and s is a nonzero real value. In these applications, attention is focused on positive solutions. We provide a simple characterization of existence and uniqueness of positive solutions when b is positive and A is irreducible.

Keywords. fixed points, spectral radius, concavity, convexity, economics

### 1. INTRODUCTION

The system of equations

$$x = Ax + b$$
 where A is a linear map (1)

occurs naturally in many areas of mathematics and statistics and its properties are well-known. Often these properties connect to the spectral radius r(A) of the operator A. For example, when x takes values in  $\mathbb{R}^n$ , A is irreducible and b is nonnegative and nonzero, the system (1) has a unique everywhere positive solution if and only if r(A) < 1. (See, for example, [3], Theorem 3.2, or [20], Theorem 2.3.6.)

In this paper, rather than (1), we study the transformed system

$$x = (Ax^{s})^{1/s} + b, (2)$$

where x is a real-valued function, powers are taken pointwise, and  $s \in \mathbb{R}$  is nonzero. Such systems arise in a range of economic and financial problems, as well as in discrete time dynamic optimization. In these cases, the interest is typically on positive solutions, since they relate to prices or physical quantities. Our aim is to provide

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necessary and sufficient conditions for existence and uniqueness of these positive solutions. Our main result shows the following under regularity conditions that are always satisfied for the finite-dimensional case: for a positive irreducible linear operator A and positive vector b, the system (2) has a unique (strictly) positive solution if and only if  $r(A)^{1/s} < 1$ . This generalizes the result mentioned in the first paragraph for the affine system (1).

In order to obtain the result stated above, our first step is to extend a pair of fixed point theorems originally obtained in [8], which consider order-preserving concave and convex operators that map an order interval of an ordered Banach space into itself. We extend these results to order-preserving operators acting either on the positive cone or the interior of the positive cone (rather than an order interval). With these extended fixed point results we prove the sufficiency component of our main theorem (i.e., conditions under which  $r(A)^{1/s} < 1$  is sufficient for existence and uniqueness of solutions to (2)). We prove necessity using an argument that builds on the Krein– Rutman theorem. Both our sufficient and our necessary conditions are valid for all nonzero values of s.

We give several applications of our result. The first looks at solutions to recursive utility models, focusing on the class of Epstein–Zin preference settings that have become increasingly important for specifying and solving asset pricing models. The next considers state-dependent discounting in an optimal savings problem used in macroeconomic modeling. The third examines the wealth-consumption ratio and the fourth considers output growth with CES production technology.

Regarding related work, studies that directly tackle existence and uniqueness of solutions to (2) are scarce. However, if we write (2) as x = Bx + b where *B* is allowed to be nonlinear, then we obtain a generalization of operator equations studied by [23, 24, 6, 25]. In this line of work, *B* is typically assumed to be  $\alpha$ -concave. (An operator *B* is  $\alpha$ -concave if for any  $t \in (0, 1)$ , there exists  $\alpha \in (0, 1)$  such that  $B(tx) \ge t^{\alpha}Bx$ .). This property is not generally satisfied in our case. Alternatively, Theorem 2.6 of [23] provides sufficient conditions for existence and uniqueness of positive solutions when *B* is positive homogeneous of degree one on the positive cone. This result is also not directly applicable to our problem.

A second major line of work related to our results involves analysis using the Thompson and Hilbert projective metrics. One connection is the following: under the conditions considered in this paper, the map associated with (2) is order-preserving and subhomogeneous. This implies nonexpansiveness in the Thompson metric (see, e.g.,

[16]). The monograph [19] provides foundational theory for nonexpansive mappings under these metrics, including conditions under which local stability of locally differentiable mappings implies global stability. (Local stability requires that the spectral radius of the gradient evaluated at the fixed point is less than one.) The results in [19] are extended to semidifferentiable maps in [1]. Other work that gives related conditions for subhomogeneous maps includes [14], who requires the fixed point be locally attractive, and [15], who requires a stronger version of the order-preserving property.

Although the results from the previous paragraph can be applied to handle the powertransformed affine system (2), since the associated map  $Fx := (Ax^s)^{1/s} + b$  is both differentiable and subhomogeneous, we have chosen to build instead on fixed point theory for order-preserving concave and convex operators. (These are the results in [8] mentioned above.) The reason we have done so is that the results from the previous paragraph assume the existence of a fixed point, whereas our aim is to give exact necessary and sufficient conditions for both existence and uniqueness of positive solutions. As we show below, the map F is always either concave or convex, depending on the value of s.

Concluding our discussion of related work, we note that the results in [8] are part of a broader literature providing conditions for existence and uniqueness of fixed points of order-preserving concave and convex operators on ordered vector space. Other examples include [2, 3, 11, 13, 16]. However, none of these fixed point theorems are directly applicable in our case. For example, [3] and [11] provide results that are useful for establishing uniqueness but not existence. [16] provide a result establishing both existence and uniqueness for concave order-preserving maps, but they require order theoretic conditions (either Dedekind completeness or  $\sigma$ -order continuity plus  $\sigma$ -Dedekind completeness) that fail in our setting (unless we restrict attention to the finite-dimensional case). Their concavity condition also fails for some values of s.

## 2. Fixed Point Theory

Let  $(B, \|\cdot\|, \leq)$  be a Banach lattice with positive cone P (see, e.g., [18]), so that  $x \leq y$  if and only if  $y - x \in P$ . We assume throughout that P is solid (i.e., has nonempty interior), and denote the interior of P by  $\mathring{P}$ . For  $x, y \in B$ , we write  $x \ll y$  if  $y - x \in \mathring{P}$ . The expression  $x \ll y$  will sometimes be written  $y \gg x$  with identical meaning. Obviously (a)  $y \in \mathring{P}$  if and only if  $0 \ll y$  and (b)  $x \ll y$  implies  $x \leq y$ . In what follows we call elements of  $\mathring{P}$  strictly positive.

Let *E* be an arbitrary subset of *B*. A function  $G: E \to B$  is called *order-preserving* on *E* if  $x, y \in E$  and  $x \leq y$  implies  $Gx \leq Gy$ . When *E* is convex, the function *G* is called *convex* on *E* if  $G(\lambda x + (1 - \lambda)y) \leq \lambda Gx + (1 - \lambda)Gy$  for all  $x, y \in E$  and  $\lambda$  in [0, 1]. *G* is called *concave* on *E* if -G is convex on *E*. We call *G* a self-map on *E* if  $x \in E$  implies  $Gx \in E$ . We define a self-map *G* on *E* to be *globally stable* on *E* if *G* has a unique fixed point  $\bar{x}$  in *E* and, for each  $x \in E$ , we have  $||G^k x - \bar{x}|| \to 0$  as  $k \to \infty$ .

We note the following elementary properties of B.

**Lemma 2.1.** For elements  $x, y, z \in B$ , the following statements are true:

- (i)  $x \ge 0$  and  $y \gg 0$  implies  $x + y \gg 0$ .
- (ii)  $x \ge y$  and  $y \gg z$  implies  $x \gg z$ .
- (iii)  $x \gg y$  and  $y \ge z$  implies  $x \gg z$ .
- (iv)  $\mathring{P}$  is a sublattice of B; that is,  $x, y \gg 0$  implies  $x \wedge y \gg 0$  and  $x \vee y \gg 0$ .

For completeness, a proof of Lemma 2.1 is given in the appendix.

We can now state our first fixed point result.

**Theorem 2.2** (Concave Case 1). Let G be an order-preserving concave self-map on  $\mathring{P}$ . If

- (i) for all  $x \gg 0$ , there exists a  $p \gg 0$  such that  $p \leq x$  and  $Gp \gg p$ , and
- (ii) for all  $x \gg 0$ , there exists a  $q \gg 0$  such that  $x \leq q$  and  $Gq \leq q$ ,

then G is globally stable on  $\check{P}$ .

*Proof.* Fix  $x \in \mathring{P}$ , which is nonempty by assumption. Choose  $p, q \in \mathring{P}$  as in (i)–(ii) above. Evidently G is an order-preserving concave self-map on [p,q]. By these facts, the condition  $Gp \gg p$  and Theorem 3.1 of [8], G has a unique fixed point  $\bar{x}$  in [p,q]. Since  $0 \ll p \leqslant \bar{x}$ , we have  $\bar{x} \in \mathring{P}$  by Lemma 2.1.

Now pick any  $y \in \mathring{P}$ . We claim that  $G^k y \to \bar{x}$  as  $k \to \infty$ . To see this, we use Lemma 2.1 to obtain  $y \wedge \bar{x} \gg 0$ , which means we can find  $c \gg 0$  such that  $Gc \gg c$  and  $c \leq y \wedge \bar{x}$ . In addition,  $y \vee \bar{x} \gg 0$ , so we can take  $d \gg 0$  such that  $Gd \leq d$  and  $y \vee \bar{x} \leq d$ . As a result,

$$c \leq y \wedge \overline{x} \leq y, \overline{x} \leq y \vee \overline{x} \leq d.$$

In particular, both y and  $\bar{x}$  lie in [c,d]. Moreover, applying the same result of [8] again, we see that  $\bar{x}$  is the only fixed point of G in [c,d] and, moreover,  $G^k y \to \bar{x}$  as  $k \to \infty$ .

We have now shown that  $\bar{x}$  is a fixed point of G in  $\mathring{P}$  and  $G^k y \to \bar{x}$  for all  $y \in \mathring{P}$ . This implies that  $\bar{x}$  is the unique fixed point of G in  $\mathring{P}$ . In particular, G is globally stable on  $\mathring{P}$ .

In some instances G is defined over all of the positive cone P and we are concerned about properties of G on the boundary of the cone. The next result provides conditions on G that extend global stability to all of  $P \setminus \{0\}$ .

**Theorem 2.3** (Concave Case 2). Let G be an order-preserving concave self-map on P such that conditions (i)–(ii) from Theorem 2.2 hold. If, in addition, for each nonzero  $x \in P$ , there exists an  $m \in \mathbb{N}$  such that  $G^m x \gg 0$ , then G is globally stable on  $P \setminus \{0\}$ .

*Proof.* First we note that G maps  $\mathring{P}$  to itself. To see this, fix  $x \in \mathring{P}$  and use (i) to choose a  $p \gg 0$  with  $p \leq x$  and  $Gp \gg p$ . By isotonicity of G we have  $Gx \geq Gp$ . Since  $Gp \geq p \gg 0$ , it follows that  $Gx \in \mathring{P}$ . In view of Theorem 2.2, G is globally stable when restricted to  $\mathring{P}$ , with unique fixed point  $\bar{x}$ .

Now fix nonzero  $x \in P$ . Our proof will be complete if we can find an  $\alpha \in (0, 1)$  and an  $M < \infty$  such that  $\|G^k x - \bar{x}\| \leq \alpha^k M$  for all  $k \in \mathbb{N}$ .

By hypothesis, we can choose an  $m \in \mathbb{N}$  such that  $x_m := G^m x \gg 0$ . Also, since G is globally stable when restricted to  $\mathring{P}$ , we can take a  $\beta \in (0, 1)$  and  $N < \infty$  such that  $||G^k x_m - \bar{x}|| \leq \beta^k N$  for all  $k \in \mathbb{N}$ . Let  $e_i = ||G^i x - x_m||$  for  $i = 1, \ldots, m$  and let  $e = \max_{i \leq m} e_i$ . If we set  $\alpha := \beta$  and  $M := \beta^{-m} \max\{N, e\}$ , it is easy to verify that  $||G^k x - \bar{x}|| \leq \alpha^k M$  for all  $k \in \mathbb{N}$ . We conclude that G is globally stable on  $P \setminus \{0\}$ , as claimed.

Next we turn to the convex case.

**Theorem 2.4** (Convex Case). Let G be an order-preserving convex self-map on the positive cone P. Suppose that

(i)  $G0 \gg 0$  and (ii) for all  $x \in \mathring{P}$ , there exists  $a \ b \in P$  such that  $x \leq b$  and  $Gb \ll b$ .

If these two conditions hold, then G is globally stable on  $\mathring{P}$ .

*Proof.* Fix  $x \in \mathring{P}$  and choose  $b \in P$  as in (ii) above. Since G is an order-preserving convex self-map on [0, b] and  $Gb \ll b$ , Theorem 3.1 of [8] implies that G has a unique

fixed point  $\bar{x}$  in [0, b]. Moreover,  $0 \leq \bar{x}$ , so isotonicity of G implies  $0 \ll G0 \leq G\bar{x} = \bar{x}$ . Hence  $\bar{x} \in \mathring{P}$ .

Now pick any  $y \in \mathring{P}$ . We claim that  $G^k y \to \bar{x}$ . Since  $y \vee \bar{x} \in \mathring{P}$ , so we can take  $d \in P$  such that  $Gd \ll d$  and  $y \vee \bar{x} \leq d$ . Note that  $0 \leq y, \bar{x} \leq y \vee \bar{x} \leq d$ . In particular, y and  $\bar{x}$  are in [0, d]. Applying the same result of [8] again, this time to G on [0, d], we see that  $\bar{x}$  is the only fixed point of G in [0, d] and, moreover,  $G^k y \to \bar{x}$ . This last result also implies that  $\bar{x}$  is the only fixed point of G in [0, d] and, moreover,  $G^k y \to \bar{x}$ . This last result the proof is now done.

### 3. Power-Transformed Affine Systems

In this section we apply the fixed point results of Section 2 to the class of nonlinear equations discussed in the introduction. Below, in Section 4, we will discuss how these problems arise in applications. Here we focus on characterizing existence and uniqueness of positive solutions.

3.1. Environment. Returning to the system (2), we assume throughout that the unknown object x takes values in the set C(T) of continuous real-valued functions on compact Hausdorff space T, and that A is a linear operator from C(T) to itself. The set C(T) is a Banach lattice when paired with the supremum norm and the usual pointwise partial order. In line with Section 2, we let P be the positive cone of C(T) and  $\mathring{P}$  be the interior of P. An *ideal* in C(T) is a vector subspace  $I \subset C(T)$  such that  $g \in I$  and  $f \in C(T)$  with  $|f| \leq |g|$  implies  $f \in I$ .

As in the introduction, powers are pointwise operations, so that  $x^s$  is the function  $T \ni t \mapsto x^s(t)$ . For the powers in (2) to make sense we operate only on positive elements of C(T), as clarified below. Obviously, our treatment includes the finitedimensional case. (If T is finite with n elements, then we endow T with the discrete topology, in which case C(T) is isometrically isomorphic to  $\mathbb{R}^n$  and (2) can be viewed as a system of n equations in  $\mathbb{R}^n$ .)

A linear operator  $A: C(T) \to C(T)$  is called *positive* if A is a self-map on P and *power compact* if there exists a  $k \in \mathbb{N}$  such that  $A^k$  is compact (i.e., maps the unit ball of C(T) to a relatively compact subset of C(T)). A positive linear operator A is called *irreducible* if the only nontrivial closed ideal on which A is invariant is the whole space C(T). The spectral radius r(A) of A can be defined by Gelfand's formula  $r(A) = \lim_{k\to\infty} ||A^k||^{1/k}$ , where  $||\cdot||$  is the operator norm on the bounded linear operators from C(T) to itself.

3.2. Equivalency. Let  $A: C(T) \to C(T)$  be a positive linear operator and suppose  $b \in C(T)$  and  $s \in \mathbb{R}$  with  $s \neq 0$  and  $b \gg 0$ . In studying (2), we also handle the representation

$$y = ((Ay)^{1/s} + b)^s, (3)$$

which is obtained from (2) via the change of variable  $y = x^s$ . The system (3) is often easier to handle than the original system (2), largely because the transformation on the right is decomposed into one purely linear operation (i.e.,  $y \mapsto Ay$ ) and one nonlinear operation.

We solve the power-transformed affine systems (2) and (3) by converting them into fixed point problems associated with the self-maps F, G on  $\mathring{P}$  defined by

$$Fx = (Ax^{s})^{1/s} + b$$
 and  $Gy = ((Ay)^{1/s} + b)^{s}$ 

Let H be the homeomorphism from  $\mathring{P}$  to itself defined by  $Hx = x^s$ . Then F and G are topologically conjugate under H, in the sense that  $H \circ F = G \circ H$ . As a result, F has a unique fixed point in  $\mathring{P}$  if and only if the same is true for G, and, moreover, fixed points of F and G have the same stability properties.

3.3. **Positive Solutions.** In what follows, if  $x \in P$  and x satisfies (2) then we call x a *positive* solution to (2). If, in addition,  $x \gg 0$ , then we call x a *strictly positive* solution. Analogous terminology is used for solutions to (3).

**Theorem 3.1.** Suppose  $b \gg 0$  and  $s \in \mathbb{R}$  with  $s \neq 0$ . If a positive operator A is irreducible and power compact, then the following statements are equivalent:

(i)  $r(A)^{1/s} < 1$ . (ii) G is globally stable on  $\mathring{P}$ .

Moreover, if  $r(A)^{1/s} \ge 1$ , then G has no fixed point in  $\mathring{P}$ .

As an immediate consequence, we have

**Corollary 3.2.** Under the conditions on A, b and s stated in Theorem 3.1, both (2) and (3) have a unique strictly positive solution if and only if  $r(A)^{1/s} < 1$ .

**Remark 3.1.** Since A is irreducible we have r(A) > 0 (see, e.g., [18], Lemma 4.2.9). Hence we can rewrite  $r(A)^{1/s} < 1$  as  $(1/s) \cdot \ln r(A) < 0$ . Thus, condition (ii) is equivalent to the statement that  $r(A) \neq 1$  and s and  $\ln r(A)$  have opposite signs. 3.4. **Proof of Theorem 3.1.** We assume throughout that the conditions on *A*, *s* and *b* imposed in Theorem 3.1 hold.

**Proposition 3.3.** *G* is order-preserving on  $\mathring{P}$  and has the following shape properties:

- (i) G is concave on  $\mathring{P}$  whenever  $s \in \mathbb{R} \setminus [0, 1)$ .
- (ii) G is convex on  $\mathring{P}$  whenever  $s \in (0, 1]$ .

*Proof.* Fix  $t \in T$  and define

$$\varphi_t(r) := \left[ r^{1/s} + b(t) \right]^s \qquad (r > 0).$$

Then, for any  $y \in \mathring{P}$ , (Gy)(t) can be written as  $\varphi_t[(Ay)(t)]$ .

We first show that G is order-preserving. Since A is linear, irreducible (and hence positive), A is order-preserving on C(T). It is easy to see that  $\varphi_t$  is an increasing function for all  $t \in T$ , so  $t \in T$  and  $y, z \in \mathring{P}$  with  $y \leq z$  implies

$$(Gy)(t) = \varphi_t[(Ay)(t)] \leq \varphi_t[(Az)(t)] = (Gz)(t).$$

In particular, G is order preserving on  $\mathring{P}$ .

To see that G is concave on  $\mathring{P}$  when s < 0 or  $s \ge 1$ , we fix  $y, z \in \mathring{P}$  and  $\lambda \in [0, 1]$  and let  $h := \lambda y + (1 - \lambda)z$ . Observe that  $\varphi_t$  is concave in this case for all  $t \in T$ . Hence, fixing  $t \in T$  and using linearity of A,

$$\varphi_t[(Ah)(t)] = \varphi_t[\lambda(Ay)(t) + (1 - \lambda)(Az)(t)]$$
  
$$\geq \lambda \varphi_t[(Ay)(t)] + (1 - \lambda)\varphi_t[(Az)(t)].$$

Hence  $(Gh)(t) \ge \lambda(Gy)(t) + (1 - \lambda)(Gz)(t)$ . Since  $t \in T$  was arbitrary, we conclude that G is concave on  $\mathring{P}$ .

Similarly, (ii) follows from the fact that each  $\varphi_t$  is convex when  $s \in (0, 1]$ .

**Proposition 3.4.** If G has a fixed point in  $\mathring{P}$ , then  $r(A)^{1/s} < 1$ .

*Proof.* Let  $\mathcal{M}$  be the topological dual space for C(T). Let  $A^*$  be the adjoint operator associated with A.<sup>1</sup> Since A is irreducible and power compact, Lemma 4.2.11 of [18] ensures us existence of an  $e^* \in \mathcal{M}$  such that

$$\langle e^*, x \rangle > 0 \text{ for all } x \gg 0 \text{ and } A^* e^* = r(A)e^*.$$
 (4)

<sup>&</sup>lt;sup>1</sup>The adjoint of a bounded linear operator  $A : C(T) \to C(T)$  is given by  $A^* : \mathcal{M} \to \mathcal{M}$  such that  $\langle A^* \mu, x \rangle = \langle \mu, Ax \rangle$  for all  $x \in C(T)$  and all  $\mu \in \mathcal{M}$ .

Let  $\bar{y}$  be a fixed point of G in  $\check{P}$ .

First consider the case where s > 0. In this case we have  $\varphi_t[y(t)] > y(t)$  for all  $y \in \mathring{P}$ and  $t \in T$ , so

$$\overline{y}(t) = (G\overline{y})(t) = \varphi_t[(A\overline{y})(t)] > (A\overline{y})(t)$$

Hence  $\bar{y} \gg A\bar{y}$ . Taking  $e^*$  as in (4), we then have  $\langle e^*, A\bar{y} - \bar{y} \rangle < 0$ , or, equivalently,  $\langle e^*, A\bar{y} \rangle < \langle e^*, \bar{y} \rangle$ . Using the definition of the adjoint and (4) gives  $r(A)\langle e^*, \bar{y} \rangle = \langle A^*e^*, \bar{y} \rangle = \langle e^*, A\bar{y} \rangle$ , so it must be that  $r(A)\langle e^*, \bar{y} \rangle < \langle e^*, \bar{y} \rangle$ . As a result, r(A) < 1. Because s > 0, we have  $r(A)^{1/s} < 1$ .

Now consider the case where s < 0. We have  $\varphi_t[y(t)] < y(t)$  for all  $y \in \mathring{P}$  and  $t \in T$ . As a result,

$$\overline{y}(t) = (G\overline{y})(t) = \varphi_t[(A\overline{y})(t)] < (A\overline{y})(t).$$

But then, taking  $e^*$  as in (4), we have  $\langle e^*, A\bar{y} - \bar{y} \rangle > 0$ , or, equivalently,  $\langle e^*, A\bar{y} \rangle > \langle e^*, \bar{y} \rangle$ . Using (4) again gives  $r(A)\langle e^*, \bar{y} \rangle = \langle e^*, A\bar{y} \rangle$ , so  $r(A)\langle e^*, \bar{y} \rangle > \langle e^*, \bar{y} \rangle$ . Hence r(A) > 1. Because s < 0, this yields  $r(A)^{1/s} < 1$ .

**Proposition 3.5.** If  $r(A)^{1/s} < 1$ , then, for each  $y \in \mathring{P}$ , there exists a pair  $p, q \in \mathring{P}$  such that  $p \leq y \leq q$ ,  $Gp \gg p$  and  $Gq \ll q$ .

In the proof of Proposition 3.5, we set r := r(A). We use the fact that r > 0, and that there exists a dominant eigenvector  $e \in \mathring{P}$  such that Ae = r(A)e, as follows from irreducibility and eventual compactness of A (see, e.g., Lemma 4.2.14 of [18].)

*Proof.* Observe that, for each  $c \in (0, \infty)$ , we have

$$\frac{G(ce)}{ce} = \left(\frac{(cre)^{1/s} + b}{(ce)^{1/s}}\right)^s = \left(r^{1/s} + \frac{b}{(ce)^{1/s}}\right)^s.$$

Now consider the case where s < 0 and r > 1. Recall that T is compact and  $b \gg 0$ . Then  $c \to 0$  implies that  $(G(ce))/(ce) \to r$  uniformly, which in turn implies

$$\exists c_0 > 0 \text{ and } \delta_0 > 1 \text{ such that } c \leq c_0 \implies G(ce) \geq \delta_0 ce.$$
(5)

Also, if  $c \to \infty$ , then  $(G(ce))/(ce) \to 0$  uniformly, which in turn implies

$$\exists c_1 > 0 \text{ and } \delta_1 < 1 \text{ such that } c \ge c_1 \implies G(ce) \le \delta_1 ce.$$
(6)

Now suppose s > 0 and r < 1. In this case,  $c \to 0$  implies that  $(G(ce))/(ce) \to \infty$  uniformly, and hence (5) holds. Also, in the same setting,  $c \to \infty$  implies  $(G(ce))/(ce) \to r$  uniformly, and since r < 1 we have (6).

So far we have shown that (5) and (6) are both valid when  $r^s < 1$ . Now fix  $y \in \mathring{P}$ . Since  $e, y \gg 0$  and T is compact, we have  $ce \ll y$  for all sufficiently small  $c \in (0, \infty)$ . Similarly,  $ce \gg y$  for all sufficiently large c. Combining these facts with (5) and (6), we obtain  $p, q \in \mathring{P}$  with  $p \leq y \leq q$ ,  $Gp \gg p$  and  $Gq \ll q$ .

Proof of Theorem 3.1. Let the conditions of Theorem 3.1 hold, in the sense that  $b \gg 0, s \in \mathbb{R} \setminus \{0\}$  and A is irreducible and power compact. If  $s \in (0, 1]$  and r(A) < 1, then G is order-preserving and convex on  $\mathring{P}$  by Proposition 3.3. Moreover,  $G0 = b^s \gg 0$  and, by Proposition 3.5, for each  $y \in \mathring{P}$  there is a  $q \in \mathring{P}$  such that  $y \leq q$  and  $q \ll Gq$ . Hence, by Theorem 2.4, G is globally stable on  $\mathring{P}$ .

If, on the other hand, s < 0 and r(A) > 1 or  $s \ge 1$  and r(A) < 1, then G is orderpreserving and concave on  $\mathring{P}$  by Proposition 3.3. Moreover, by Proposition 3.5, for each  $y \in \mathring{P}$  there is a pair  $p, q \in \mathring{P}$  such that  $p \le y \le q$ ,  $Gp \gg p$  and  $q \ll Gq$ . Hence, by Theorem 2.2, G is globally stable on  $\mathring{P}$ .

We have now shown that  $r(A)^{1/s} < 1$  implies G is globally stable on  $\mathring{P}$ . For the converse we apply Proposition 3.4, which tells us that G has no fixed point in  $\mathring{P}$  when  $r(A)^{1/s} \ge 1$ .

#### 4. Applications

In this section we give applications of the preceding results on power-transformed affine systems.

4.1. **Recursive Preferences.** In finance, Epstein–Zin preferences have become increasingly important for specifying and solving mainstream asset pricing models (see, e.g., [17, 12, 4, 21, 10, 7]). Similar specifications have also been adopted in macroe-conomics (see, e.g., [5]). A generic specification takes the form

$$\nu = ((1 - \beta)c^{\rho} + \beta(R\nu)^{\rho})^{1/\rho}$$
(7)

where  $c \in \mathring{P} \subset C(T)$  denotes consumption in each state and  $\rho$  and  $\beta$  are positive parameters. The operator R is defined at  $v \in \mathring{P}$  by  $Rv = (Qv^{\alpha})^{1/\alpha}$ , where  $\alpha$  is a nonzero parameter and Q is an irreducible Markov operator. The unknown function v gives lifetime utility at each state of world.

Equation (7) has a unique positive solution if and only if  $\beta < 1$ . This fact can be derived from our results by setting  $w = v^{\alpha}$ ,  $s = \alpha/\rho$ , and rewriting (7) as

$$w = \left\{ (1-\beta)c^{\rho} + \beta(Qw)^{\rho/\alpha} \right\}^{\alpha/\rho} = \left\{ (1-\beta)c^{\rho} + (\beta^{s}Qw)^{1/s} \right\}^{s}.$$

By Corollary 3.2, a unique strictly positive solution exists if and only if the linear operator  $A := \beta^s Q$  is such that  $r(A)^{1/s} < 1$ . Since Q is a Markov operator we have r(Q) = 1, and hence  $r(A)^{1/s} < 1$  if and only if  $\beta < 1$ .

4.2. State-Dependent Discounting. [22] studies an optimal consumption problem and shows that optimal consumption from current wealth w takes the form

$$c(z) = b(z)^{-1/\gamma} w$$

where  $\gamma > 0$  is a parameter and b is a function of an uncertainty state  $z \in Z$  satisfying the equation

$$b(z) = \left\{ 1 + \left[ \beta(z) R(z)^{1-\gamma} (Qb)(z) \right]^{1/\gamma} \right\}^{\gamma}.$$
 (8)

Here  $\beta(z) > 0$  is the discount factor in state z, R(z) > 0 is the gross interest rate in state z and Qb is defined at each  $z \in Z$  by

$$(Qb)(z) = \sum_{z' \in Z} b(z')q(z,z'),$$

where q is the transition matrix of an irreducible Markov chain. [22] shows that (8) has a strictly positive solution if and only if the spectral radius of the linear operator A defined by

$$(Af)(z) := \beta(z)R(z)^{1-\gamma} \sum_{z' \in Z} f(z')q(z,z')$$
(9)

has a spectral radius less than unity. In the setting of [22], the set Z is finite.

We can prove the same result using Corollary 3.2, while also adding uniqueness. Indeed, with A defined as in (9), we can express (8) as

$$b(z) = \left\{ 1 + \left[ (Ab)(z) \right]^{1/\gamma} \right\}^{\gamma}.$$
 (10)

Since q is irreducible and  $\beta$  and R are strictly positive, the operator A is irreducible. Eventual compactness of A is immediate from finiteness of Z. Hence Corollary 3.2 implies that (10) has a unique strictly positive solution if and only if  $r(A)^{1/\gamma} < 1$ . Since  $\gamma > 0$ , this reduces to r(A) < 1, which is the same condition used by [22].

Notice that Corollary 3.2 can also be used to relax the finiteness assumption on Z by assuming instead that Z is a compact Hausdorff space. In this case continuity restrictions need to be placed on  $\beta$  and R.

4.3. Wealth-Consumption Ratio. Another important object in asset pricing is the wealth-consumption ratio of the representative agent. In the setting of Section 4.1, the equilibrium wealth-consumption ratio w under Epstein-Zin preferences satisfies the first-order condition

$$\beta^s Q w^s = (w-1)^s, \tag{11}$$

where the operator Q is assumed to be an irreducible and power compact linear operator that contains information about consumption growth. Letting  $A := \beta^{s} Q$ , we can rewrite (11) as

$$w = (Aw^s)^{1/s} + 1,$$

which has the same form as (2). By Corollary 3.2, a unique strictly positive wealthconsumption ratio exists if and only if the linear operator A satisfies  $r(A)^{1/s} < 1$ , or equivalently,  $\beta r(Q)^{1/s} < 1$ .

4.4. Growth with CES Production Function. Consider a discrete time multisector growth model with total depreciation and no population growth or technological progress. The law of motion for the capital-labor ratio is given by  $k_{t+1} = sf(Ak_t)$ , where *s* is a savings rate, *f* is a CES production function, *A* is an irreducible matrix that characterizes the technology, and  $k_t$  is a vector of multi-sector capital-labor ratios. Then the steady state capital-labor ratio satisfies

$$k = s \{\theta + (1 - \theta)(Ak)^{\rho}\}^{1/\rho}, \qquad (12)$$

where  $\theta \in (0, 1)$ ,  $\rho \neq 0$ , and all algebraic operations are performed elementwise. We can rewrite (12) as

$$k = \left\{ s^{\rho}\theta + \left( s(1-\theta)^{1/\rho}Ak \right)^{\rho} \right\}^{1/\rho}$$

Then by Corollary 3.2, a unique and strictly positive vector of multi-sector capitallabor ratios exists in the steady state if and only if  $s^{\rho}(1-\theta)r(A) < 1$ .

## APPENDIX A. APPENDIX

Let B' be the topological dual space of B and take P' to be the set of all positive linear functionals on B; that is, all  $x' \in B'$  such that  $\langle x', x \rangle \ge 0$  for all  $x \in P$ . The next lemma follows from Corollary 2.8 of [9]:

**Lemma A.1.** In our setting, the following statements are equivalent:

(i) x ≫ 0.
(ii) ⟨x', x⟩ > 0 for all nonzero x' ∈ P'.

 $(iii) \cup_{n \in \mathbb{N}} [-nx, nx] = B.$ 

Using Lemma A.1, we can verify Lemma 2.1.

*Proof of Lemma 2.1.* Regarding (i), fix  $x, y \in B$  with  $x \ge 0$  and  $y \ge 0$ . Fix nonzero  $x' \in P'$ . Applying Lemma A.1, we have  $\langle x', x + y \rangle = \langle x', x \rangle + \langle x', y \rangle > 0$ . Hence  $x \ge 0$ .

Regarding (ii), given nonzero  $x' \in P'$ , we have x = (x - y) + (y - z) and  $x \gg 0$  follows from (i). The proof of (iii) is similar.

For the final claim, fix  $x, y \gg 0$ . Since  $x \lor y \ge x$ , the supremum is strictly positive. For the case of the infimum, let  $z = x \land y$  and pick any  $b \in B$ . By Lemma A.1, for n sufficiently large, we have  $-nx \le b \le nx$  and  $-ny \le b \le ny$ . It follows directly that  $b \le (nx) \land (ny) = n(x \land y) = nz$ . Also,  $nx, ny \ge -b$ , so  $(nx) \land (ny) \ge -b$ , or  $-nz \le b$ . In particular, there exists an  $n \in \mathbb{N}$  with  $-nz \le b \le nz$ . Applying Lemma A.1 yields  $z \gg 0$ .

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