Unique Solutions to Power-Transformed Affine Systems

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Abstract. Systems of the form $x = (Ax^s)^{1/s} + b$ arise in a range of economic, financial and control problems, where A is a linear operator acting on a space of real-valued functions (or vectors) and s is a nonzero real value. In these applications, attention is focused on positive solutions. We provide a simple and complete characterization of existence and uniqueness of positive solutions under conditions on A and b that imply positivity.

1. Introduction

The system of equations

$$x = Ax + b \quad \text{where } A \text{ is a linear map} \tag{1}$$

occurs naturally in many areas of mathematics and statistics. For example, an x solving (1) is a steady state for the elementary vector difference equation $x_{t+1} = Ax_t + b$. As a second example, consider the discrete Lyapunov equation from control

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theory, which takes the form $GXG^\top - X + Q = 0$, where all elements are matrices and X is the unknown. Since $X \mapsto GXG^\top$ is linear, this is also a version of (1).

In this paper, we study the transformed system

$$x = (Ax^s)^{1/s} + b, \qquad s \in \mathbb{R} \text{ and } s \neq 0, \tag{2}$$

where powers are taken pointwise (e.g., if $x = (x_i) \in \mathbb{R}^n$, then $x^s = (x_i^s) \in \mathbb{R}^n$). Such systems arise in a range of economic and financial problems, as well as in discrete time dynamic optimization. In these cases, the interest is typically on positive solutions, since they relate to prices or physical quantities. (Moreover, focusing on positive solutions avoids having to handle fractional powers of negative numbers.)

In this paper we provide a complete characterization of existence and uniqueness of positive solutions under conditions on A and b that imply positivity. In particular, we show that, under a regularity condition that is always satisfied for the finite-dimensional case, the system (2) has a unique (strictly) positive solution if and only if $r(A)^s < 1$, where $r(A)$ is the spectral radius of A. This finding is a natural extension of the observation that, when A is irreducible and b is positive, (1) has a unique positive solution if and only if $r(A) < 1$.

We obtain the results stated above by applying a fixed point theorem for positive concave (or convex) operators on the positive cone. This fixed point theorem are a small extension of known results originally obtained by Du (1990). A range of similar fixed point results based on monotonicity and order concavity (or convexity) are also relevant (see, e.g., Amann, 1972, 1976; Krasnosel'skii et al., 2012; Guo and Lakshmikantham, 1988, etc.), although they are not directly applicable in our case.[1] Our work is also related to Marinacci and Montrucchio (2019), who study Tarski-type fixed points of monotone operators with applications in recursive utilities, as we do. However, they do not study the particular class of equations that we do (i.e., those of type (2)). Marinacci and Montrucchio (2010) and Borovička and Stachurski (2020) provide related results on existence and uniqueness of recursive utilities.

Studies that directly tackle (2) are scarce. However, if we write (2) as $x = Bx+b$ where B is allowed to be nonlinear, then we obtain a generalization of operator equations

[1] Uniqueness of a positive solution can be guaranteed since the right hand side of (2) is increasing and strongly sublinear (Guo and Lakshmikantham, 1988, Theorem 2.2.2). However, existence usually requires additional assumptions (see, e.g., Krasnosel'skii et al. (2012)).

studied by Zhai et al. (2008, 2010); Berzig and Samet (2013); Zhai and Wang (2017), among others. In this line of work, B is typically assumed to be α-concave[2], a property not satisfied in our case when $Bx = (Ax^s)^{1/s}$. To the best of our knowledge, the most closely related result in this literature is Theorem 2.6 of Zhai et al. (2008), which provides sufficient conditions for existence and uniqueness of positive solutions when B is homogeneous. (Despite its generality, they make assumptions on the operator B that are not applicable to our problem.)

In all of the work mentioned above, only sufficient conditions are provided. In contrast, our aim is to give exact necessary and sufficient conditions for existence and uniqueness of positive solutions.

2. Fixed Point Theory

Let $(B, \|\cdot\|, \leqslant)$ be a Banach lattice with positive cone P (see, e.g., Meyer-Nieberg (2012)). By definition, we have $x \leqslant y$ if and only if $y - x \in P$. We assume throughout that P is solid (i.e., has nonempty interior), and denote the interior of P by $\mathring{P}$. For $x, y \in B$, we write $x \ll y$ if $y - x \in \mathring{P}$. The expression $x \ll y$ will sometimes be written $y \gg x$ with identical meaning. Obviously (a) $y \in \mathring{P}$ if and only if $0 \ll y$ and (b) $x \ll y$ implies $x \leqslant y$. In what follows we call elements of $\mathring{P}$ *strictly positive.*

Let E be an arbitrary subset of B. A function $G \colon E \to B$ is called *order-preserving* on E if $x, y \in E$ and $x \leqslant y$ implies $Gx \leqslant Gy$. When E is convex, the function G is called *convex* on E if $G(\lambda x + (1-\lambda)y) \leqslant \lambda Gx + (1-\lambda)Gy$ for all $x, y \in E$ and λ in $[0,1]$. G is called *concave* on E if $-G$ is convex on E. We call G a self-map on E if $x \in E$ implies $Gx \in E$. We define a self-map G on E to be *globally stable* on E if G has a unique fixed point $\bar{x}$ in E and, for each $x \in E$, we have $\|G^k x - \bar{x}\| \to 0$ as $k \to \infty$.

We let B' be the topological dual space of B and take P' to be the set of all positive linear functionals on B; that is, all $x' \in B'$ such that $\langle x', x \rangle \geqslant 0$ for all $x \in P$. The next lemma follows from Corollary 2.8 of Glueck and Weber (2020):

Lemma 2.1. *In our setting, the following statements are equivalent:*

(i) $x \gg 0$.

[2]An operator B is α-concave if for any $t \in (0,1)$, there exists $\alpha \in (0,1)$ such that $B(tx) \geqslant t^\alpha Bx$.

(ii) $\langle x', x \rangle > 0$ *for all nonzero* $x' \in P'$.

(iii) $\cup_{n \in \mathbb{N}} [-nx, nx] = B$.

Lemma 2.1 is useful for establishing basic properties of $\mathring{P}$. For example,

Lemma 2.2. *For elements* $x, y, z \in B$*, the following statements are true:*

(i) $x \geqslant 0$ *and* $y \gg 0$ *implies* $x + y \gg 0$.

(ii) $x \geqslant y$ *and* $y \gg z$ *implies* $x \gg z$.

(iii) $x \gg y$ *and* $y \geqslant z$ *implies* $x \gg z$.

(iv) $\mathring{P}$ *is a sublattice of* B*; that is,* $x, y \gg 0$ *implies* $x \wedge y \gg 0$ *and* $x \vee y \gg 0$.

Proof. Regarding (i), fix $x, y \in B$ with $x \geqslant 0$ and $y \gg 0$. Fix nonzero $x' \in P'$. Applying Lemma 2.1, we have $\langle x', x + y \rangle = \langle x', x \rangle + \langle x', y \rangle > 0$. Hence $x \gg 0$.

Regarding (ii), given nonzero $x' \in P'$, we have $x = (x - y) + (y - z)$ and $x \gg 0$ follows from (i). The proof of (iii) is similar.

For the final claim, fix $x, y \gg 0$. Since $x \vee y \geqslant x$, the supremum is strictly positive. For the case of the infimum, let $z = x \wedge y$ and pick any $b \in B$. By Lemma 2.1, for n sufficiently large, we have $-nx \leqslant b \leqslant nx$ and $-ny \leqslant b \leqslant ny$. It follows directly that $b \leqslant (nx) \wedge (ny) = n(x \wedge y) = nz$. Also, $nx, ny \geqslant -b$, so $(nx) \wedge (ny) \geqslant -b$, or $-nz \leqslant b$. In particular, there exists an $n \in \mathbb{N}$ with $-nz \leqslant b \leqslant nz$. Applying Lemma 2.1 yields $z \gg 0$. □

Theorem 2.3. *Let* G *be an order-preserving concave self-map on* $\mathring{P}$*. If*

(i) for all $x \gg 0$*, there exists a* $p \gg 0$ *such that* $p \leqslant x$ *and* $Gp \gg p$*, and*

(ii) for all $x \gg 0$*, there exists a* $q \gg 0$ *such that* $x \leqslant q$ *and* $Gq \leqslant q$*,*

then G *is globally stable on* $\mathring{P}$.

Proof. Fix $x \in \mathring{P}$, which is nonempty by assumption. Choose $p, q \in \mathring{P}$ as in (i)–(ii) above. Evidently G is an order-preserving concave self-map on $[p, q]$. By these facts, the condition $Gp \gg p$ and Theorem 3.1 of Du (1990), G has a unique fixed point $\bar{x}$ in $[p, q]$. Since $0 \ll p \leqslant \bar{x}$, we have $\bar{x} \in \mathring{P}$ by Lemma 2.2.

Now pick any $y \in \mathring{P}$. We claim that $G^k y \to \bar{x}$ as $k \to \infty$. To see this, we use Lemma 2.2 to obtain $y \wedge \bar{x} \gg 0$, which means we can find $c \gg 0$ such that $Gc \gg c$

and $c \leqslant y \wedge \bar{x}$. In addition, $y \vee \bar{x} \gg 0$, so we can take $d \gg 0$ such that $Gd \leqslant d$ and $y \vee \bar{x} \leqslant d$. As a result,

$$c \leqslant y \wedge \bar{x} \leqslant y, \bar{x} \leqslant y \vee \bar{x} \leqslant d.$$

In particular, both y and $\bar{x}$ lie in $[c, d]$. Moreover, applying the same result of Du (1990) again, we see that $\bar{x}$ is the only fixed point of G in $[c, d]$ and, moreover, $G^k y \to \bar{x}$ as $k \to \infty$.

We have now shown that $\bar{x}$ is a fixed point of G in $\mathring{P}$ and $G^k y \to \bar{x}$ for all $y \in \mathring{P}$. This implies that $\bar{x}$ is the unique fixed point of G in $\mathring{P}$. In particular, G is globally stable on $\mathring{P}$. □

In some instances G is defined over all of the positive cone P and we are concerned about properties of G on the boundary of the cone. The next result provides conditions on G that extend global stability to all of $P \setminus \{0\}$.

Theorem 2.4. *Let G be an order-preserving concave self-map on P such that conditions (i)–(ii) from Theorem 2.3 hold. If, in addition, for each nonzero $x \in P$, there exists an $m \in \mathbb{N}$ such that $G^m x \gg 0$, then G is globally stable on $P \setminus \{0\}$.*

Proof. First we note that G maps $\mathring{P}$ to itself. To see this, fix $x \in \mathring{P}$ and use (i) to choose a $p \gg 0$ with $p \leqslant x$ and $Gp \gg p$. By isotonicity of G we have $Gx \geqslant Gp$. Since $Gp \geqslant p \gg 0$, it follows that $Gx \in \mathring{P}$. In view of Theorem 2.3, G is globally stable when restricted to $\mathring{P}$, with unique fixed point $\bar{x}$.

Now fix nonzero $x \in P$. Our proof will be complete if we can find an $\alpha \in (0, 1)$ and an $M < \infty$ such that $\|G^k x - \bar{x}\| \leqslant \alpha^k M$ for all $k \in \mathbb{N}$.

By hypothesis, we can choose an $m \in \mathbb{N}$ such that $x_m := G^m x \gg 0$. Also, since G is globally stable when restricted to $\mathring{P}$, we can take a $\beta \in (0, 1)$ and $N < \infty$ such that $\|G^k x_m - \bar{x}\| \leqslant \beta^k N$ for all $k \in \mathbb{N}$. Let $e_i = \|G^i x - x_m\|$ for $i = 1, \ldots, m$ and let $e = \max_{i \leqslant m} e_i$. If we set $\alpha := \beta$ and $M := \beta^{-m} \max\{N, e\}$, it is easy to verify that $\|G^k x - \bar{x}\| \leqslant \alpha^k M$ for all $k \in \mathbb{N}$. We conclude that G is globally stable on $P \setminus \{0\}$, as claimed. □

Next we turn to the convex case.

Theorem 2.5. *Let G be an order-preserving convex self-map on the positive cone P. Suppose that* *(i)* $G0 \gg 0$ *and*

(ii) for all $x \in \mathring{P}$*, there exists a* $b \in P$ *such that* $x \leqslant b$ *and* $Gb \ll b$*.*

If these two conditions hold, then G *is globally stable on* $\mathring{P}$*.*

Proof. Fix $x \in \mathring{P}$ and choose $b \in P$ as in (ii) above. Since G is an order-preserving convex self-map on $[0, b]$ and $Gb \ll b$, Theorem 3.1 of Du (1990) implies that G has a unique fixed point $\bar{x}$ in $[0, b]$. Moreover, $0 \leqslant \bar{x}$, so isotonicity of G implies $0 \ll G0 \leqslant G\bar{x} = \bar{x}$. Hence $\bar{x} \in \mathring{P}$.

Now pick any $y \in \mathring{P}$. We claim that $G^k y \to \bar{x}$. Since $y \vee \bar{x} \in \mathring{P}$, so we can take $d \in P$ such that $Gd \ll d$ and $y \vee \bar{x} \leqslant d$. Note that $0 \leqslant y, \bar{x} \leqslant y \vee \bar{x} \leqslant d$. In particular, y and $\bar{x}$ are in $[0, d]$. Applying the same result of Du (1990) again, this time to G on $[0, d]$, we see that $\bar{x}$ is the only fixed point of G in $[0, d]$ and, moreover, $G^k y \to \bar{x}$. This last result also implies that $\bar{x}$ is the only fixed point of G in $\mathring{P}$. Evidently G maps $\mathring{P}$ to itself, so the proof is now done. □

3. Power-Transformed Affine Systems

In this section we apply the fixed point results of Section 2 to the class of nonlinear equations discussed in the introduction. Below, in Section 4, we will discuss how these problems arise in applications. Here we focus on characterizing existence and uniqueness of positive solutions.

3.1. Environment.

Returning to the system (2), we assume throughout that the unknown object x takes values in the set $C(T)$ of continuous real-valued functions on compact Hausdorff space T, and that A is a linear operator from $C(T)$ to itself. The set $C(T)$ is a Banach lattice when paired with the supremum norm and the usual pointwise partial order. In line with Section 2, we let P be the positive cone of $C(T)$ and $\mathring{P}$ be the interior of P. An *ideal* in $C(T)$ is a vector subspace $I \subset C(T)$ such that $g \in I$ and $f \in C(T)$ with $|f| \leqslant |g|$ implies $f \in I$.

As in the introduction, powers are pointwise operations, so that x^s is the function $T \ni t \mapsto x^s(t)$. For the powers in (2) to make sense we operate only on positive elements of $C(T)$, as clarified below. Obviously, our treatment includes the finite-dimensional case. (If T is finite with n elements, then we endow T with the discrete topology, in which case $C(T)$ is isometrically isomorphic to $\mathbb{R}^n$ and (2) can be viewed as a system of n equations in $\mathbb{R}^n$.)

A linear operator $A\colon C(T) \to C(T)$ is called *positive* if A is a self-map on P and *eventually compact* if there exists a $k \in \mathbb{N}$ such that A^k is compact (i.e., maps the unit ball of $C(T)$ to a relatively compact subset of $C(T)$). A positive linear operator A is called *irreducible* if the only nontrivial closed ideal on which A is invariant is the whole space $C(T)$. The spectral radius $r(A)$ of A can be defined by Gelfand's formula $r(A) = \lim_{k\to\infty} \|A^k\|^{1/k}$, where $\|\cdot\|$ is the operator norm on the bounded linear operators from $C(T)$ to itself.

3.2. **Equivalency.** Let $A\colon C(T) \to C(T)$ be a positive linear operator and suppose $b \in C(T)$ and $s \in \mathbb{R}$ with $s \neq 0$ and $b \gg 0$. In studying (2), we also handle the representation

$$y = ((Ay)^{1/s} + b)^s, \tag{3}$$

which is obtained from (2) via the change of variable $y = x^s$. The system (3) is often easier to handle than the original system (2), largely because the transformation on the right is decomposed into one purely linear operation (i.e., $y \mapsto Ay$) and one nonlinear operation.

We solve the power-transformed affine systems (2) and (3) by converting them into fixed point problems associated with the self-maps F, G on $\mathring{P}$ defined by

$$Fx = (Ax^s)^{1/s} + b \quad \text{and} \quad Gy = ((Ay)^{1/s} + b)^s$$

Let H be the homeomorphism from $\mathring{P}$ to itself defined by $Hx = x^s$. Then F and G are topologically conjugate under H, in the sense that $H \circ F = G \circ H$. As a result, F has a unique fixed point in $\mathring{P}$ if and only if the same is true for G, and, moreover, fixed points of F and G have the same stability properties.

3.3. **Positive Solutions.** In what follows, if $x \in P$ and x satisfies (2) then we call x a *positive* solution to (2). If, in addition, $x \gg 0$, then we call x a *strictly positive* solution. Analogous terminology is used for solutions to (3).

Theorem 3.1. *Suppose $b \gg 0$ and $s \in \mathbb{R}$ with $s \neq 0$. If A is irreducible and eventually compact, then the following statements are equivalent:*

(i) $r(A)^s < 1$.

(ii) G is globally stable on $\mathring{P}$.

Moreover, if $r(A)^s \geqslant 1$, then G has no fixed point in $\mathring{P}$

As an immediate consequence, we have

Corollary 3.2. *Under the conditions on A, b and s stated in Theorem 3.1, both (2) and (3) have a unique strictly positive solution if and only if $r(A)^s < 1$.*

Remark 3.1. Since A is irreducible we have $r(A) > 0$ (see, e.g., Meyer-Nieberg (2012), Lemma 4.2.9). Hence we can rewrite $r(A)^s < 1$ as $s \cdot \ln r(A) < 0$. Thus, condition (ii) is equivalent to the statement that $r(A) \neq 1$ and s and $\ln r(A)$ have opposite signs.

3.4. **Proof of Theorem 3.1.** We assume throughout that the conditions on A, s and b imposed in Theorem 3.1 hold.

Proposition 3.3. *G is order-preserving on $\mathring{P}$ and has the following shape properties:*

(i) G is concave on $\mathring{P}$ whenever $s \in \mathbb{R} \setminus [0, 1)$.
(ii) G is convex on $\mathring{P}$ whenever $s \in (0, 1]$.

Proof. Fix $t \in T$ and define

$$\varphi_t(r) := \left[r^{1/s} + b(t)\right]^s \qquad (r > 0).$$

Then, for any $y \in \mathring{P}$, $(Gy)(t)$ can be written as $\varphi_t[(Ay)(t)]$.

We first show that G is order-preserving. Since A is linear, irreducible (and hence positive), A is order-preserving on $C(T)$. It is easy to see that φ_t is an increasing function for all $t \in T$, so $t \in T$ and $y, z \in \mathring{P}$ with $y \leqslant z$ implies

$$(Gy)(t) = \varphi_t[(Ay)(t)] \leqslant \varphi_t[(Az)(t)] = (Gz)(t).$$

In particular, G is order preserving on $\mathring{P}$.

To see that G is concave on $\mathring{P}$ when $s < 0$ or $s \geqslant 1$, we fix $y, z \in \mathring{P}$ and $\lambda \in [0, 1]$ and let $h := \lambda y + (1 - \lambda)z$. Observe that φ_t is concave in this case for all $t \in T$. Hence, fixing $t \in T$ and using linearity of A,

$$\begin{aligned}\varphi_t[(Ah)(t)] &= \varphi_t[\lambda(Ay)(t) + (1 - \lambda)(Az)(t)] \\ &\geqslant \lambda\varphi_t[(Ay)(t)] + (1 - \lambda)\varphi_t[(Az)(t)].\end{aligned}$$

Hence $(Gh)(t) \geqslant \lambda(Gy)(t) + (1-\lambda)(Gz)(t)$. Since $t \in T$ was arbitrary, we conclude that G is concave on $\mathring{P}$.

Similarly, (ii) follows from the fact that each φ_t is convex when $s \in (0, 1]$. □

Proposition 3.4. *If G has a fixed point in $\mathring{P}$, then $r(A)^s < 1$.*

Proof. Let $\mathcal{M}$ be the topological dual space for $C(T)$. Let A^* be the adjoint operator associated with A.[3] Since A is irreducible and eventually compact, Lemma 4.2.11 of Meyer-Nieberg (2012) ensures us existence of an $e^* \in \mathcal{M}$ such that

$$\langle e^*, x \rangle > 0 \text{ for all } x \gg 0 \quad \text{and} \quad A^* e^* = r(A) e^*. \tag{4}$$

Let $\bar{y}$ be a fixed point of G in $\mathring{P}$.

First consider the case where $s > 0$. In this case we have $\varphi_t[y(t)] > y(t)$ for all $y \in \mathring{P}$ and $t \in T$, so

$$\bar{y}(t) = (G\bar{y})(t) = \varphi_t[(A\bar{y})(t)] > (A\bar{y})(t).$$

Hence $\bar{y} \gg A\bar{y}$. Taking e^* as in (4), we then have $\langle e^*, A\bar{y} - \bar{y} \rangle < 0$, or, equivalently, $\langle e^*, A\bar{y} \rangle < \langle e^*, \bar{y} \rangle$. Using the definition of the adjoint and (4) gives $r(A)\langle e^*, \bar{y} \rangle = \langle A^* e^*, \bar{y} \rangle = \langle e^*, A\bar{y} \rangle$, so it must be that $r(A)\langle e^*, \bar{y} \rangle < \langle e^*, \bar{y} \rangle$. As a result, $r(A) < 1$. Because $s > 0$, we have $r(A)^s < 1$.

Now consider the case where $s < 0$. We have $\varphi_t[y(t)] < y(t)$ for all $y \in \mathring{P}$ and $t \in T$. As a result,

$$\bar{y}(t) = (G\bar{y})(t) = \varphi_t[(A\bar{y})(t)] < (A\bar{y})(t).$$

But then, taking e^* as in (4), we have $\langle e^*, A\bar{y} - \bar{y} \rangle > 0$, or, equivalently, $\langle e^*, A\bar{y} \rangle > \langle e^*, \bar{y} \rangle$. Using (4) again gives $r(A)\langle e^*, \bar{y} \rangle = \langle e^*, A\bar{y} \rangle$, so $r(A)\langle e^*, \bar{y} \rangle > \langle e^*, \bar{y} \rangle$. Hence $r(A) > 1$. Because $s < 0$, this yields $r(A)^s < 1$. □

Proposition 3.5. *If $r(A)^s < 1$, then, for each $y \in \mathring{P}$, there exists a pair $p, q \in \mathring{P}$ such that $p \leqslant y \leqslant q$, $Gp \gg p$ and $Gq \ll q$.*

In the proof of Proposition 3.5, we set $r := r(A)$. We use the fact that $r > 0$, and that there eixsts a dominant eigenvector $e \in \mathring{P}$ such that $Ae = r(A)e$, as follows from irreducibility and eventual compactness of A (see, e.g., Lemma 4.2.14 of Meyer-Nieberg (2012).)

[3] The adjoint of a bounded linear operator $A : C(T) \to C(T)$ is given by $A^* : \mathcal{M} \to \mathcal{M}$ such that $\langle A^* \mu, x \rangle = \langle \mu, Ax \rangle$ for all $x \in C(T)$ and all $\mu \in \mathcal{M}$.

Proof. Observe that, for each $c \in (0, \infty)$, we have

$$\frac{G(ce)}{ce} = \left(\frac{(cre)^{1/s} + b}{(ce)^{1/s}}\right)^s = \left(r^{1/s} + \frac{b}{(ce)^{1/s}}\right)^s.$$

Now consider the case where $s < 0$ and $r > 1$. Recall that T is compact and $b \gg 0$. Then $c \to 0$ implies that $(G(ce))/(ce) \to r$ uniformly, which in turn implies

$$\exists\, c_0 > 0 \text{ and } \delta_0 > 1 \text{ such that } c \leqslant c_0 \implies G(ce) \geqslant \delta_0 ce. \tag{5}$$

Also, if $c \to \infty$, then $(G(ce))/(ce) \to 0$ uniformly, which in turn implies

$$\exists\, c_1 > 0 \text{ and } \delta_1 < 1 \text{ such that } c \geqslant c_1 \implies G(ce) \leqslant \delta_1 ce. \tag{6}$$

Now suppose $s > 0$ and $r < 1$. In this case, $c \to 0$ implies that $(G(ce))/(ce) \to \infty$ uniformly, and hence (5) holds. Also, in the same setting, $c \to \infty$ implies $(G(ce))/(ce) \to r$ uniformly, and since $r < 1$ we have (6).

So far we have shown that (5) and (6) are both valid when $r^s < 1$. Now fix $y \in \mathring{P}$. Since $e, y \gg 0$ and T is compact, we have $ce \ll y$ for all sufficiently small $c \in (0, \infty)$. Similarly, $ce \gg y$ for all sufficiently large c. Combining these facts with (5) and (6), we obtain $p, q \in \mathring{P}$ with $p \leqslant y \leqslant q$, $Gp \gg p$ and $Gq \ll q$. □

Proof of Theorem 3.1. Let the conditions of Theorem 3.1 hold, in the sense that $b \gg 0$, $s \in \mathbb{R} \setminus \{0\}$ and A is irreducible and eventually compact. If $s \in (0, 1]$ and $r(A) < 1$, then G is order-preserving and convex on $\mathring{P}$ by Proposition 3.3. Moreover, $G0 = b^s \gg 0$ and, by Proposition 3.5, for each $y \in \mathring{P}$ there is a $q \in \mathring{P}$ such that $y \leqslant q$ and $q \ll Gq$. Hence, by Theorem 2.5, G is globally stable on $\mathring{P}$.

If, on the other hand, $s < 0$ and $r(A) > 1$ or $s \geqslant 1$ and $r(A) < 1$, then G is order-preserving and concave on $\mathring{P}$ by Proposition 3.3. Moreover, by Proposition 3.5, for each $y \in \mathring{P}$ there is a pair $p, q \in \mathring{P}$ such that $p \leqslant y \leqslant q$, $Gp \gg p$ and $q \ll Gq$. Hence, by Theorem 2.3, G is globally stable on $\mathring{P}$.

We have now shown that $r(A)^s < 1$ implies G is globally stable on $\mathring{P}$. For the converse we apply Proposition 3.4, which tells us that G has no fixed point in $\mathring{P}$ when $r(A)^s \geqslant 1$. □

4. Applications

In this section we give several applications of the preceding results on power-transformed affine systems.

4.1. **State-Dependent Discounting.** Toda (2019) studies an optimal consumption problem and shows that optimal consumption from current wealth w takes the form

$$c(z) = b(z)^{-1/\gamma} w$$

where $\gamma > 0$ is a parameter and b is a function of an uncertainty state $z \in Z$ satisfying the equation

$$b(z) = \left\{1 + \left[\beta(z) R(z)^{1-\gamma} (Qb)(z)\right]^{1/\gamma}\right\}^{\gamma}. \tag{7}$$

Here $\beta(z) > 0$ is the discount factor in state z, $R(z) > 0$ is the gross interest rate in state z and Qb is defined at each $z \in Z$ by

$$(Qb)(z) = \sum_{z' \in Z} b(z') q(z, z'),$$

where q is the transition matrix of an irreducible Markov chain. Toda (2019) shows that (7) has a strictly positive solution if and only if the spectral radius of the linear operator A defined by

$$(Af)(z) := \beta(z) R(z)^{1-\gamma} \sum_{z' \in Z} f(z') q(z, z') \tag{8}$$

has a spectral radius less than unity. In the setting of Toda (2019), the set Z is finite.

We can prove the same result using Corollary 3.2, while also adding uniqueness. Indeed, with A defined as in (8), we can express (7) as

$$b(z) = \left\{1 + [(Ab)(z)]^{1/\gamma}\right\}^{\gamma}. \tag{9}$$

Since q is irreducible and β and R are strictly positive, the operator A is irreducible. Eventual compactness of A is immediate from finiteness of Z. Hence Corollary 3.2 implies that (9) has a unique strictly positive solution if and only if $r(A)^{\gamma} < 1$. Since $\gamma > 0$, this reduces to $r(A) < 1$, which is the same condition used by Toda (2019).

Notice that Corollary 3.2 can also be used to relax the finiteness assumption on Z by assuming instead that Z is a compact Hausdorff space. In this case continuity restrictions need to be placed on β and R.

4.2. **Recursive Preferences.** In finance, Epstein–Zin preferences have become increasingly important for specifying and solving mainstream asset pricing models (see, e.g., Hansen and Scheinkman (2012), Bansal et al. (2012), Schorfheide et al. (2018) or Gomez-Cram and Yaron (2021)). Similar specifications have also been adopted in macroeconomics (see, e.g., Basu and Bundick (2017)). A generic specification takes the form

$$v = ((1-\beta)c^\rho + \beta(Rv)^\rho)^{1/\rho} \tag{10}$$

where $c \in \mathring{P} \subset C(T)$ denotes consumption in each state and ρ and β are positive parameters. The operator R is defined at $v \in \mathring{P}$ by $Rv = (Qv^\alpha)^{1/\alpha}$, where α is a nonzero parameter and Q is an irreducible Markov operator. The unknown function v gives lifetime utility at each state of world.

It is known (see, e.g., Borovička and Stachurski (2020)) that (10) has a unique positive solution if and only if $\beta < 1$. This result can be derived from our results by setting $w = v^\alpha$, $s = \alpha/\rho$, and rewriting (10) as

$$w = \left\{(1-\beta)c^\rho + \beta(Qw)^{\rho/\alpha}\right\}^{\alpha/\rho} = \left\{(1-\beta)c^\rho + (\beta^s Qw)^{1/s}\right\}^s.$$

By Corollary 3.2, a unique strictly positive solution exists if and only if the linear operator $A := \beta^s Q$ is such that $r(A)^s < 1$, which can also be written as $r(A)^{1/s} < 1$. Since Q is a Markov operator we have $r(Q) = 1$, and hence $r(A)^{1/s} < 1$ if and only if $\beta < 1$.

4.3. **Wealth-Consumption Ratio.** Another important object in asset pricing is the wealth-consumption ratio of the representative agent. In a similar environment to the one described above, the equilibrium wealth-consumption ratio w under Epstein-Zin preferences satisfies the following first-order condition

$$\beta^s Q w^s = (w-1)^s, \tag{11}$$

where the operator Q is assumed to be an irreducible and eventually compact linear operator that contains information about consumption growth. Letting $A := \beta^s Q$, we can rewrite (11) as

$$w = (Aw^s)^{1/s} + 1,$$

which has the same form as (2). By Corollary 3.2, a unique strictly positive wealth-consumption ratio exists if and only if the linear operator A satisfies $r(A)^s < 1$, or equivalently, $\beta r(Q)^{1/s} < 1$.

4.4. **Growth with CES Production Function.** Consider a discrete time multi-sector growth model with total depreciation and no population growth or technological progress. The law of motion for the capital-labor ratio is given by $k_{t+1} = sf(Ak_t)$, where s is a savings rate, f is a CES production function, A is an irreducible matrix that characterizes the technology, and k_t is a vector of multi-sector capital-labor ratios. Then the steady state capital-labor ratio satisfies

$$k = s\left\{\theta + (1-\theta)(Ak)^\rho\right\}^{1/\rho}, \tag{12}$$

where $\theta \in (0,1)$, $\rho \neq 0$, and all algebraic operations are performed elementwise. We can rewrite (12) as

$$k = \left\{s^\rho\theta + \left(s(1-\theta)^{1/\rho}Ak\right)^\rho\right\}^{1/\rho}.$$

Then by Corollary 3.2, a unique and strictly positive vector of multi-sector capital-labor ratios exists in the steady state if and only if $s^\rho(1-\theta)r(A) < 1$.

References

Amann, H. (1972): "On the number of solutions of nonlinear equations in ordered Banach spaces," *Journal of functional analysis*, 11, 346–384.

——— (1976): "Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces," *SIAM review*, 18, 620–709.

Bansal, R., D. Kiku, and A. Yaron (2012): "An Empirical Evaluation of the Long-Run Risks Model for Asset Prices," *Critical Finance Review*, 1, 183–221.

Basu, S. and B. Bundick (2017): "Uncertainty shocks in a model of effective demand," *Econometrica*, 85, 937–958.

Berzig, M. and B. Samet (2013): "Positive fixed points for a class of nonlinear operators and applications," *Positivity*, 17, 235–255.

Borovička, J. and J. Stachurski (2020): "Necessary and sufficient conditions for existence and uniqueness of recursive utilities," *The Journal of Finance*, 75, 1457–1493.

Du, Y. (1990): "Fixed points of increasing operators in ordered Banach spaces and applications," *Applicable Analysis*, 38, 1–20.

Glueck, J. and M. R. Weber (2020): "Almost interior points in ordered Banach spaces and the long-term behaviour of strongly positive operator semigroups," *Studia mathematica*, 254, 237–263.

Gomez-Cram, R. and A. Yaron (2021): "How important are inflation expectations for the nominal yield curve?" *The Review of Financial Studies*, 34, 985–1045.

Guo, D. and V. Lakshmikantham (1988): *Nonlinear problems in abstract cones*, Academic Press.

Hansen, L. P. and J. A. Scheinkman (2012): "Recursive utility in a Markov environment with stochastic growth," *Proceedings of the National Academy of Sciences*, 109, 11967–11972.

Krasnosel'skii, M., G. Vainikko, R. Zabreyko, Y. Ruticki, and V. Stet'senko (2012): *Approximate Solution of Operator Equations*, Springer Netherlands.

Marinacci, M. and L. Montrucchio (2010): "Unique solutions for stochastic recursive utilities," *Journal of Economic Theory*, 145, 1776–1804.

——— (2019): "Unique tarski fixed points," *Mathematics of Operations Research*, 44, 1174–1191.

Meyer-Nieberg, P. (2012): *Banach lattices*, Springer Science & Business Media.

Schorfheide, F., D. Song, and A. Yaron (2018): "Identifying long-run risks: A Bayesian mixed-frequency approach," *Econometrica*, 86, 617–654.

Toda, A. A. (2019): "Wealth distribution with random discount factors," *Journal of Monetary Economics*, 104, 101–113.

Zhai, C. and L. Wang (2017): "φ-(h, e)-concave operators and applications," *Journal of Mathematical Analysis and Applications*, 454, 571–584.

Zhai, C.-B., C. Yang, and C.-M. Guo (2008): "Positive solutions of operator equations on ordered Banach spaces and applications," *Computers & Mathematics with Applications*, 56, 3150–3156.

Zhai, C.-B., C. Yang, and X.-Q. Zhang (2010): "Positive solutions for nonlinear operator equations and several classes of applications," *Mathematische Zeitschrift*, 266, 43–63.