Asset Pricing with Time Preference Shocks: Existence and Uniqueness

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Abstract. This paper studies existence and uniqueness of recursive utility in asset pricing models with time preference shocks. We provide conditions that clarify existence and uniqueness for a wide range of models, including exact necessary and sufficient conditions for standard formulations. The conditions isolate the roles of preference parameters, as well as the different risks that drive the consumption and preference shock processes. By deriving and decomposing a stability coefficient for recursive utility models, we show how different parameters in the model interact to determine existence and uniqueness of solutions.

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1. Introduction

Models that combine recursive preferences with time preference shocks form one of the key approaches to reconciling observed asset price dynamics with time paths for dividends and other cash flows. For example, Albuquerque et al. (2016) show that time preference shocks provide a solution to the correlation puzzle, while Schorfheide et al. (2018) argue

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that time preference shocks are crucial to match the dynamics of the risk-free rate. Gomez-Cram and Yaron (2020) highlight the importance of time preference shocks to explain the variation in nominal bond yields. Further examples of this rapidly growing literature can be found in Basu and Bundick (2017), Chen and Yang (2019), de Groot et al. (2018), Creal and Wu (2020), and de Groot et al. (2022). In all of this work, recursive preferences are formulated via the Epstein–Zin specification.

Surprisingly, the question of whether or not the models described above have well-defined solutions at the stated parameterizations has remained open until now. This paper provides conditions to analyze existence and uniqueness for asset pricing models with time preference shocks that are general enough to cover a wide range of models, including all of those listed above. Moreover, for standard formulations of the time preference shock we provide exact necessary and sufficient conditions. In addition, we show that when the conditions are violated, the models make no usable predictions (i.e., existence fails).

To motivate our results, we first connect recursive utility to the wealth-consumption ratio, which is central to the analysis of prices and returns (since it controls the stochastic discount factor that maps cash flows to prices). We show that recursive utility exists if and only if the wealth-consumption ratio is positive and finite. We then show that our existence and uniqueness condition is equivalent to a strictly positive minimum discount rate. This provides a natural economic interpretation of our results.

In addition, we show that the delineation between existence and nonexistence depends on the spectral radius of a valuation operator embedding three Epstein–Zin preference parameters. To more easily connect our main theoretical results to applications, we use a local spectral radius theorem to prove that, when time preference shocks and consumption shocks are independent (which holds in many applications), the stability coefficient can be decomposed into three terms. These terms depend on the rate of time preference, the time path for time preference shocks and the dynamics of consumption growth respectively. In this manner, we are able to isolate the roles of time preferences, time preference shocks and consumption dynamics.

Our main findings from this exercise can be divided into three cases. First, when the intertemporal elasticity of substitution parameter ψ obeys $\psi > 1$, the lifetime utility of risk averse agents is suppressed in the presence of time preference shocks, making the existence of a solution more likely. Intuitively, this is because, after adjusting for intertemporal elasticity of substitution, preference shocks can be seen as adding volatility to consumption. Since risk averse agents dislike volatile consumption flows, nontrivial time preference shocks suppress the wealth-consumption ratio, which acts against divergence. We also show that, for common calibrations of models with time preference shocks and $\psi > 1$, as in, say, Albuquerque et al. (2016), Schorfheide et al. (2018) and Gomez-Cram and Yaron (2020), the influence of preference shocks on existence is large and significantly dominates consumption risk.

Second, for the case $\psi < 1$, which is adopted in Campbell (1996), Basu and Bundick (2017), and other studies, we show that time preference shocks increase the likelihood that no stable solution exists, and that small increases in the persistence or the volatility of any state process can change a model from having a well-defined solution to one where no solution exists. This is because the income effect dominates the substitution effect, and hence agents choose a higher wealth-consumption ratio when preference shocks are added to the model.[1]

Third, for the special case $\psi = 1$, which is also adopted in many quantitative studies (see, e.g., Tallarini (2000), Hansen et al. (2008), or Wachter (2013)), we show that there always exists a unique solution as long as the subjective discount factor β is less than one. This result includes standard Epstein–Zin preferences with unit intertemporal elasticity of substitution and no time preference shocks as a special case.[2]

We also examine knife-edge properties of the model when ψ is close to one. We show that, in the standard formulation of the model of Albuquerque et al. (2016), the stability coefficient diverges to minus infinity when ψ approaches 1 from above. Hence, for ψ sufficiently close to but larger than 1, a solution always exists. In contrast, if ψ approaches 1 from below, the stability coefficient diverges to infinity and we have nonexistence. These results are in line with de Groot et al. (2018), who show that the influence of time preference shocks on model outcomes can become arbitrarily large for ψ close to one. To eliminate the asymptote, they provide an alternative utility specification, the asset pricing implications of which are discussed in de Groot et al. (2022). As an additional contribution, we provide exact necessary and sufficient conditions for existence and uniqueness under this alternative specification.

Our paper builds on a large literature that deals with existence and uniqueness of recursive utility, including Epstein and Zin (1989), Boyd (1990), Marinacci and Montrucchio (2010), Hansen and Scheinkman (2012), Bäuerle and Jaśkiewicz (2018), Becker and Rincon-Zapatero (2018a), Becker and Rincon-Zapatero (2018b), Marinacci and Montrucchio (2019),

[1] For evidence in favor of setting $\psi < 1$, see, e.g., Havranek et al. (2015) or Calvet et al. (2021).

[2] The case $\psi = 1$ is frequently seen in literature because it allows for closed-form solutions for continuation utility and bond prices under specific assumptions for the cash flow process. At the same time, no analytical solution exists under general consumption dynamics, so the existence and uniqueness of recursive utility becomes significant. Guo and He (2018) give sufficient conditions for existence and uniqueness assuming a finite state space, while Hansen and Scheinkman (2012) gives sufficient conditions for existence but does not cover uniqueness. Christensen (2022) proves existence and uniqueness under a "thin-tail" condition. We extend these results to a setting with time preference shocks.

Pohl et al. (2023), Borovička and Stachurski (2020), Balbus (2020), Bloise et al. (2021) and Christensen (2022). However, none of these papers considers time preference shocks.[3] In contrast, this paper explicitly includes time preference shocks in the utility specification and focuses directly on the connection between time preference shocks and existence and uniqueness of recursive utility. As our approach does not require strong assumptions on the underlying consumption process, it can also handle cases in which time preference shocks are added to models that rely on other risk factors, such as models with consumption disasters (e.g., Barro (2009) and Wachter (2013)), models with volatility of volatility (e.g., Bollerslev et al. (2009)), and models with jumps in volatility and growth rates (e.g., Drechsler and Yaron (2011)).

Methodologically, this paper builds on Hansen and Scheinkman (2012), Borovička and Stachurski (2020), and Stachurski and Zhang (2021). The first two papers use the spectral radius of a valuation operator to analyze existence and uniqueness of recursive utilities, while the last studies dynamic programming with state-dependent discounting. Hansen and Scheinkman (2012) and Borovička and Stachurski (2020) omit consideration of time preference shocks, whereas this paper is entirely focused on models with these shocks. In addition, while Stachurski and Zhang (2021) focus on dynamic programming and optimality conditions, here we provide new analytical tools for recursive preference models that are general enough to settle the question of existence and uniqueness for a broad range of asset pricing models with time preference shocks (while Stachurski and Zhang (2021) only provide sufficient conditions under a restricted set of parameters). Moreover, our results are based on a new fixed point theorem, which allows us to treat a range of utility specifications for Epstein–Zin utility with preference shocks, including the specification in Albuquerque et al. (2016) as well as the alternative specification proposed by de Groot et al. (2018).

Our results are also related to Pohl et al. (2023), who derive sufficient conditions for existence and nonexistence under general consumption dynamics. Instead of using the spectral radius of a valuation operator, their paper uses a concept of relative existence: by proving existence (nonexistence) for a given set of parameters, they show that existence (nonexistence) follows for certain other parameter combinations. This implies that there is also a parameter region where the method is inconclusive. Our results have the advantage of being both necessary and sufficient and they cover the whole parameter space. More importantly, Pohl et al. (2023) do not consider time preference shocks, which are the focus of this paper.

[3]While Creal and Wu (2020) analyze the effects of time preference shocks on bond prices via the log-linearization technique of Campbell and Shiller (1988), we provide general necessary and sufficient conditions for the original non-linear model. Pohl et al. (2018) highlight the importance of nonlinearities in such models.

Finally, our paper contributes to the debate on how to include time preference shocks into asset pricing models, also called "valuation risk." Kruger (2021) shows that the valuation risk model of Albuquerque et al. (2016) implies counterfactually large preferences for early resolution of uncertainty and extreme aversion to valuation risk. He also shows that if ψ approaches 1 from above, risk premia can get arbitrarily large, making the model hard to test empirically. de Groot et al. (2022) show that the way valuation risk is introduced into the Epstein–Zin utility matters, and the specification commonly used in the literature (e.g., in Albuquerque et al. (2016) and Schorfheide et al. (2018)) lacks several desirable properties of the standard Epstein–Zin preferences. They propose an alternative specification of valuation risks which does not suffer from this problem. Our framework covers and facilitates analysis of both specifications.

The rest of the paper is structured as follows. Section 2 sets out the model. Section 3 provides our main results. Section 4 decomposes the stability coefficient. Section 5 discusses applications. Section 6 studies the alternative specification proposed in de Groot et al. (2018), including the case $\psi = 1$. Section 7 concludes. Proofs are in the appendix.

2. Environment

We consider an endowment economy containing a representative agent with Epstein-Zin preferences (see Epstein and Zin (1989) and Weil (1990)). Our first set of results apply to the formulation of Albuquerque et al. (2016) and Schorfheide et al. (2018), where preferences are defined recursively by

$$V_t = \left[(1-\beta)\lambda_t C_t^{1-1/\psi} + \beta \left\{\mathcal{R}_{t,1-\gamma}\left(V_{t+1}\right)\right\}^{1-1/\psi}\right]^{1/(1-1/\psi)}. \tag{1}$$

Here

- $\{C_t\}_{t\geqslant 0}$ is a consumption path and $\beta \in (0,1)$ is a time discount factor,
- $\gamma \neq 1$ governs risk aversion and $\psi \neq 1$ is the IES without time preference shocks,
- $\{\lambda_t\}_{t\geqslant 0}$ is a sequence of time preference shocks,
- V_t is the utility value of the path extending on from time t and
- $\mathcal{R}_{t,1-\gamma}$ is the Kreps–Porteus certainty equivalent operator conditional on time t information, defined by

$$\mathcal{R}_{t,1-\gamma}(V_{t+1}) = (\mathbb{E}_t V_{t+1}^{1-\gamma})^{1/(1-\gamma)}. \tag{2}$$

The growth rates of consumption and the time preference shock process are given by

$$\ln\left(\frac{C_{t+1}}{C_t}\right) = g_c(X_t, X_{t+1}, \xi_{t+1}) \quad \text{and} \quad \ln\left(\frac{\lambda_{t+1}}{\lambda_t}\right) = g_\lambda(X_t, X_{t+1}, \xi_{t+1}), \tag{3}$$

where

- $\{X_t\}_{t \geqslant 0}$ is a discrete time Markov process on a compact metric space X,
- $\{\xi_t\}_{t \geqslant 1}$ is an IID process supported on $\mathbb{Y} \subset \mathbb{R}^k$, and
- $g_i \colon \mathsf{X} \times \mathsf{X} \times \mathbb{Y} \to \mathbb{R}$ is continuous for each $i \in \{c, \lambda\}$.

The processes $\{X_t\}$ and $\{\xi_t\}$ are assumed to be independent. We seek a solution for the utility process $\{V_t\}$.

The state process $\{X_t\}$ updates according to transition density q, in the sense that

$$\mathbb{P}\{X_{t+1} \in B \mid X_t = x\} = \int_B q(x, y) \, \mathrm{d}y \tag{4}$$

for each $x \in \mathsf{X}$ and Borel subset B of X. Let q^n denote the n-step transition density.

Assumption 2.1. The process $\{X_t\}_{t \geqslant 0}$ is stationary, with $X_t \stackrel{\mathrm{d}}{=} \pi$ for all $t \geqslant 0$. The transition density q is continuous on $\mathsf{X} \times \mathsf{X}$ and there exists an $\ell \in \mathbb{N}$ such that q^ℓ is everywhere positive.

We admit the case where X is finite, in which case the right-hand side of (4) is interpreted as $\sum_{y \in B} q(x, y)$, and q is a transition matrix. Assumption 2.1 then becomes equivalent to the statement that q is aperiodic and irreducible.[4]

Remark 2.1. The assumption that typically binds in the list above is compactness of X. At the same time, the assumption holds if we truncate innovations. The applications we consider below use standard normal innovations. When standard normal innovations are truncated to lie in the interval $[-k, k]$, the exogenous state process becomes quantitatively indistinguishable from the original whenever k is large.[5]

3. Geometric Stability

In this section we present our main results on the solution of (1).

3.1. Set Up.

To obtain a stationary solution for recursive utility, our first step is to normalize $\{V_t\}$ by dividing out the growing components $\{C_t\}$ and $\{\lambda_t\}$. This leads us to introduce the transformed variable

$$G_t := \frac{1}{\lambda_t^\theta} \left(\frac{V_t}{C_t} \right)^{1-\gamma} \quad \text{where} \quad \theta := \frac{1-\gamma}{1-1/\psi}. \tag{5}$$

[4]In the discrete setting we adopt the discrete topology, so q and all other functions on the state are automatically continuous.

[5]We omit explicit truncation during computation, since all numbers are automatically truncated to the range of 64 bit floats.

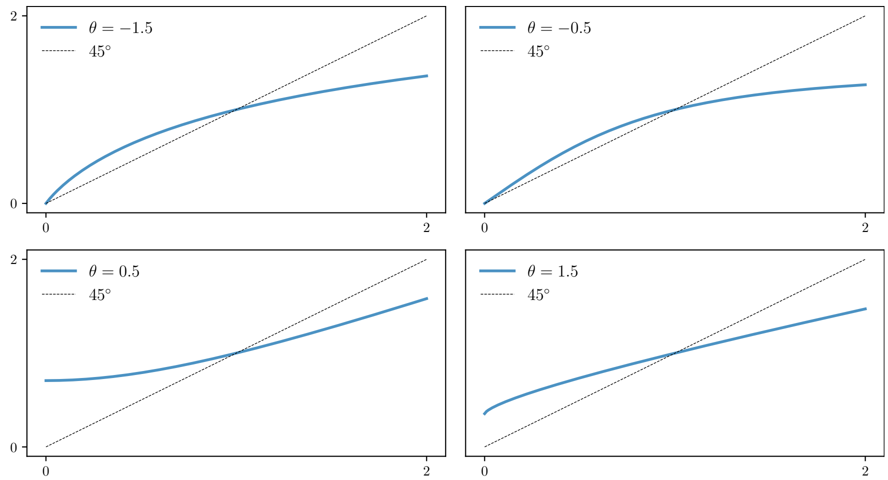

FIGURE 1. Shape properties of F.

Although this particular normalization is non-standard, it possesses one major advantage: the linear and nonlinear components of the evaluation naturally separate. To see how this works, we divide (1) by $\lambda_t^{\frac{1}{1-1/\psi}} C_t$ and raise it to the power of $1-\gamma$. By using the definition of G_t we can then rewrite (1) as

$$G_t = F\left[\,\mathbb{E}_t\, G_{t+1}\, \exp\left\{\theta g_\lambda(X_t, X_{t+1}, \xi_{t+1}) + (1-\gamma) g_c(X_t, X_{t+1}, \xi_{t+1})\right\}\,\right], \tag{6}$$

where

$$F(t) := \left(1 - \beta + \beta\, t^{1/\theta}\right)^{\theta} \qquad (t \geqslant 0).$$

If $\theta < 0$, we set $F(0) = 0$, which makes F continuous. Figure 1 shows that F is either concave increasing or convex increasing, depending on the value of θ.[6] Evidently, any solution $\{G_t\}$ to (6) yields a solution $\{V_t\}$ to the original utility problem (1), which can be obtained by reversing the transformation in (5).

3.2. **Existence and Uniqueness.** To study solutions to (6), we take $\mathscr{C}$ to be the set of continuous everywhere positive functions on X. In what follows, a *stationary Markov solution* to (6) is a $g \in \mathscr{C}$ such that the stochastic process $\{G_t\} := \{g(X_t)\}$ satisfies (6) with probability one for all $t \geqslant 0$. For any operator $\mathbb{T} \colon \mathscr{C} \to \mathscr{C}$, we call $\mathbb{T}$ *globally geometrically stable* on $\mathscr{C}$ if $\mathbb{T}$ has a unique fixed point $g^* \in \mathscr{C}$ and, for all $g \in \mathscr{C}$, there exists an $a < 1$ and $N < \infty$ such that $\sup_{x \in \mathsf{X}} |(\mathbb{T}^n g)(x) - g^*(x)| \leqslant a^n N$ for all $n \in \mathbb{N}$.

Let $\mathbb{K}$ be the linear operator defined by

$$(\mathbb{K} g)(x) = \mathbb{E}_x\left[\, g(X_{t+1})\, \Gamma(X_t, X_{t+1}, \xi_{t+1})\right] \qquad (x \in \mathsf{X}), \tag{7}$$

[6] In the figure we set $\beta = 0.5$. Any $\beta \in (0,1)$ produces the same basic shape properties.

where $\mathbb{E}_x$ conditions on $X_t = x$ and

$$\Gamma(x, y, \xi) := \exp\left\{\theta g_\lambda(x, y, \xi) + (1-\gamma) g_c(x, y, \xi)\right\}. \tag{8}$$

The operator $\mathbb{K}$ applies a form of linear discounting to rewards one period in the future. We assume that $\mathbb{E}\,\Gamma(x, y, \xi)$ is finite for all $x, y \in \mathsf{X}$.

Let $\mathbb{T}\colon \mathscr{C} \to \mathscr{C}$ be defined by

$$(\mathbb{T}g)(x) = F[(\mathbb{K}g)(x)] \qquad (g \in \mathscr{C},\ x \in \mathsf{X}). \tag{9}$$

By construction, $g \in \mathscr{C}$ is a stationary Markov solution to (6) if and only if g is a fixed point of $\mathbb{T}$. As we show in Appendix A.1, $\mathbb{T}$ is monotone increasing and either convex or concave, depending on the value of θ. As a result, we can analyze its fixed points and stability properties using the theory of monotone concave operators and monotone convex operators.[7]

The basic intuition can already be seen in Figure 1, since F has these same properties (i.e., monotonicity and convexity or concavity). The map F has a unique fixed point in the interior of its domain whenever $F(t) > t$ for small positive t and $F(t) < t$ for all sufficiently large t. It is also clear from the figures that, when these boundary conditions are satisfied, $F^n(t)$ converges to the unique fixed point as $n \to \infty$ for any $t > 0$.

While this intuition speaks only to F, the operator $\mathbb{K}$ is linear and monotone increasing, so the composition $\mathbb{T} = F \circ \mathbb{K}$ inherits the monotonicity and convexity (or concavity) properties of F. As a result, existence of a unique stationary Markov solution depends on an analogous set of boundary conditions for $\mathbb{T}$, where small functions are mapped strictly up and large functions are mapped strictly down. Whether or not these boundary conditions hold depends both on the parameters in F and the properties of the operator $\mathbb{K}$.

Regarding $\mathbb{K}$, it turns out that, to analyze these boundary conditions, it is enough to study the spectral radius of $\mathbb{K}$, which we denote by $r(\mathbb{K})$.[8] In stating our main theorem, we let

$$\mathscr{S} := \ln \beta + \frac{\ln r(\mathbb{K})}{\theta}. \tag{10}$$

The content of the next theorem is that the boundary conditions for $\mathbb{T}$ discussed above hold if and only if $\mathscr{S} < 0$.

Theorem 3.1. *If Assumption 2.1 holds, then the following statements are equivalent:*

(a) $\mathscr{S} < 0$.

[7]For an overview of the literature, see Zhang (2013), or the earlier work by Krasnosel'skiĭ (1964). The specific fixed point result that we use below is a modified version of a theorem due to Du (1990).

[8]Appendix A.1 clarifies the definition of the spectral radius used here and below.

(b) $\mathbb{T}$ *is globally geometrically stable on* $\mathscr{C}$.

Moreover, if $\mathscr{S} \geqslant 0$, *then no stationary Markov solution exists in* $\mathscr{C}$.

Theorem 3.1 provides an exact dichotomy. If $\mathscr{S} < 0$, then a unique and globally attracting stationary Markov solution g^* exists in $\mathscr{C}$. On the other hand, if $\mathscr{S} \geqslant 0$, then not only does global geometric stability fail, but existence fails, specifically.

In Section 4 we decompose $\mathscr{S}$ into different components and study their influence. In the rest of this section we discuss the economic intuition behind Theorem 3.1.

3.3. **The Wealth-Consumption Ratio.** We first study the wealth-consumption ratio, the significance of which was discussed in the introduction. In this model, the equilibrium wealth-consumption ratio $w(X_t) = W_t/C_t$ obeys

$$\beta^\theta \mathbb{E}_t \left[\left(\frac{\lambda_{t+1}}{\lambda_t} \right)^\theta \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \left(\frac{w(X_{t+1})}{w(X_t) - 1} \right)^\theta \right] = 1$$

(see, e.g., Schorfheide et al. (2018)).

Rearranging the previous expression gives

$$(w(X_t) - 1)^\theta = \beta^\theta \mathbb{E}_t \left[\left(\frac{\lambda_{t+1}}{\lambda_t} \right)^\theta \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} w(X_{t+1})^\theta \right].$$

Conditioning on $X_t = x$ and writing pointwise on X yields $w = 1 + \beta \left(\mathbb{K} w^\theta \right)^{1/\theta}$. A function $w \in \mathscr{C}$ solves this equation if and only if w is a fixed point of the operator $\mathbb{U} \colon \mathscr{C} \to \mathscr{C}$ defined by

$$(\mathbb{U} w) = 1 + \beta \left(\mathbb{K} w^\theta \right)^{1/\theta}. \tag{11}$$

By proving a topological conjugacy relationship between $\mathbb{U}$ and $\mathbb{T}$, we establish the following result.

Proposition 3.2. $\mathbb{U}$ *has the same stability properties as* $\mathbb{T}$. *In particular, when Assumption 2.1 holds,* $\mathbb{U}$ *is globally geometrically stable on* $\mathscr{C}$ *if and only if* $\mathscr{S} < 0$.

Inspecting the relationship between the wealth-consumption ratio and recursive utility also sheds light on the existence issue discussed in the previous section. For this model, the wealth-consumption ratio satisfies

$$w(X_t) = \frac{W_t}{C_t} = \frac{1}{1-\beta} g(X_t)^{1/\theta}, \tag{12}$$

or equivalently, $g(X_t) = (1-\beta)^\theta w(X_t)^\theta$. There is no solution $g \in \mathscr{C}$ if the wealth-consumption ratio diverges at some states, regardless of whether $\theta > 0$ or $\theta < 0$.[9] This observation can help explain the existence condition in the applications below, since the wealth-consumption ratio is directly affected by model parameters.

3.4. **Connection to Discount Rates.** The sign of the stability coefficient $\mathscr{S}$ in Theorem 3.1 hinges on the spectral radius of the valuation operator $\mathbb{K}$, which contains information about the dynamics of consumption growth and time preference shocks. In addition to the technical description of $r(\mathbb{K})$ provided in Appendix A.2, we show here that the stability coefficient is closely related to the discount rates for a payoff that can be traded in the economy.[10] This provides a natural economic interpretation of our results.

To begin, recall that in Section 3.3 we connected recursive utility to the wealth-consumption ratio and showed that recursive utility is positive and finite (at any state) if and only if the wealth-consumption ratio is positive and finite. We now give an alternative interpretation to this equivalence. To this end, we take $\{\Lambda_t\}$ to be a strictly positive pricing kernel process implied by the absence of arbitrage. Then, the wealth-consumption ratio is given by

$$\frac{W_t}{C_t} = \frac{P_{C,t}}{C_t} + 1 = \sum_{n=0}^{\infty} \mathbb{E}_t \left(\frac{\Lambda_{t+n} C_{t+n}}{\Lambda_t C_t} \right), \tag{13}$$

where $P_{C,t}$ is the ex-dividend price of a claim to aggregate consumption. Existence of recursive utility is then equivalent to (13) being positive and finite. If we define an operator A by

$$(Af)(X_t) := \mathbb{E}_t \left(\frac{\Lambda_{t+1} C_{t+1}}{\Lambda_t C_t} f(X_{t+1}) \right), \tag{14}$$

then equation (13) becomes $W_t/C_t = \sum_{n=0}^{\infty} A^n \mathbb{1}$ by the Markov property. This Neumann series converges if and only if $r(A) < 1$. Since the wealth-consumption ratio is finite if and only if $\mathscr{S} < 0$ by Proposition 3.2, it follows that $r(A) = e^{\mathscr{S}}$.

Under appropriate conditions, $e^{\mathscr{S}}$ is the dominant eigenvalue of the eigenvalue problem $A\varphi = \lambda\varphi$ and it characterizes the long-run growth or decay of the operator in the sense that $A^n f$ will contain a deterministic growth component $e^{n\mathscr{S}}$ (Hansen and Scheinkman, 2009). For the valuation operator A defined in (14), $(A^n f)(X_t)$ is the price of a time $t+n$ payoff after adjusting for consumption growth, so $-\mathscr{S}$ can be viewed as the discount rate for the payoff. Theorem 3.1 implies that the discount rate must be strictly positive.

[9] When $\theta < 0$, $w \to \infty$ implies $g = 0$ at some states, which means that $g \notin \mathscr{C}$. Note that $W_t/C_t \geqslant 1$ by definition, so the wealth-consumption ratio becoming too low will not affect existence.

[10] We are grateful to an anonymous referee for pointing out this connection.

4. Decomposition of the Stability Exponent

Next we show that, under an independence condition, the stability coefficient $\mathscr{S}$ can be decomposed into three terms. In stating the result, we let $\mathcal{R}_a(Y) = (\mathbb{E}Y^a)^{1/a}$ for any nonzero $a \in \mathbb{R}$ and positive random variable Y.

Proposition 4.1. *Let the conditions of Assumption 2.1 hold. If $\{C_t\}$ and $\{\lambda_t\}$ are independent, then*

$$\mathscr{S} = \ln \beta + \mathscr{S}_\lambda + \left(1 - \frac{1}{\psi}\right) \mathscr{S}_c, \tag{15}$$

where

$$\mathscr{S}_\lambda := \lim_{T \to \infty} \frac{1}{T} \ln \mathcal{R}_\theta \left(\frac{\lambda_T}{\lambda_0}\right) \quad \text{and} \quad \mathscr{S}_c := \lim_{T \to \infty} \frac{1}{T} \ln \mathcal{R}_{1-\gamma} \left(\frac{C_T}{C_0}\right). \tag{16}$$

Proposition 4.1 separates the influence of the time preference parameter β, the time preference shocks $\{\lambda_t\}$ and the consumption path $\{C_t\}$ on existence and uniqueness of recursive utility. Note that if there are no time preference shocks, then $\lambda_t = 1$ for all t and hence $\mathscr{S}_\lambda = 0$. The proof of Proposition 4.1 uses a local spectral radius theorem, which is employed to obtain an alternative representation of the spectral radius $r(\mathbb{K})$. (See Appendix A.2 for details.) In applications it will be seen that the more volatile and the larger the persistence of the time preference shock, the larger is $\mathscr{S}$.

The independence condition required by Proposition 4.1 holds in many (but not all) applications.[11] The applications we consider in this paper all satisfy the condition.[12]

Proposition 4.2. *If the growth rate of $\{\lambda_t\}$ has zero mean, then $\mathscr{S}_\lambda \cdot \theta \geqslant 0$.*

The key implication of Proposition 4.2 is that the sign of $\mathscr{S}_\lambda$ switches with θ. Assuming $\gamma > 1$, we have $\theta < 0$ if and only if $\psi > 1$. Thus, the IES is crucial for the effect of time preference shocks on the existence of a solution. More precisely, if $\psi > 1$, then $\mathscr{S}_\lambda \leqslant 0$, so adding time preference shocks always lowers the stability coefficient $\mathscr{S}$ and loosens the existence condition. If $\psi < 1$, then the reverse is true. In this case, the income effect dominates the substitution effect so adding time preference shocks increases the wealth-consumption

[11] Exceptions include the extended model in Albuquerque et al. (2016) and the model of Creal and Wu (2020).

[12] The independence condition might seem problematic in our setup, given (3), which suggests that consumption growth and time preference shocks depend on the same variables. However, in standard applications, it is commonly the case that $\{X_t\}$ and $\{\xi_t\}$ are vector-valued with at least some independent components, and consumption and the time preference shock are separated across these components (i.e., the components that drive the time preference shock are independent of the components that drive consumption).

ratio[13] and tightens the existence condition. We analyze the quantitative impact of time preference shocks in Section 5.

Proposition 4.2 has valuable practical implications. For example, in standard parameterizations of long-run risk models, $\gamma > 1$, $\psi > 1$ and thus $\theta < 0$. Proposition 4.2 then implies that $\mathscr{S}_\lambda \leqslant 0$, so if a solution for a model without time preference shocks exists, there also exists a solution for the same model with time preference shocks. Hence, one can directly use the results in, say, Hansen and Scheinkman (2012), Pohl et al. (2023), Borovička and Stachurski (2020), or Christensen (2022), to prove existence for asset pricing models with time preference shocks under the zero mean growth specification.

Conversely, to explore the asset pricing implications of these models when $\psi < 1$, one must be careful about the preference shock process, since Proposition 4.2 implies that a solution is less likely to exist when such shocks are added. In the next section, we illustrate this point with several long-run risk models from the literature.

5. Applications

Here we consider three applications that help illustrate the qualitative and quantitative effects of different model parameters and sources of risk on the existence of a solution. For this, we consider the models of Albuquerque et al. (2016), Schorfheide et al. (2018) and Gomez-Cram and Yaron (2020), treated from within a unified framework (which also nests many other long-run risk models). In this framework, log consumption growth $g_{c,t+1}$ follows

$$\begin{aligned} g_{c,t+1} &= \mu_c + z_t + \sigma_{c,t}\, \xi_{c,t+1} \\ z_{t+1} &= \rho\, z_t + \rho_\pi\, z_{\pi,t} + \sigma_{z,t}\, \eta_{t+1} \\ z_{\pi,t+1} &= \rho_{\pi\pi}\, z_{\pi,t} + \sigma_{z\pi,t}\, \eta_{\pi,t+1} \end{aligned} \tag{17}$$

where

$$\sigma_{i,t} = \varphi_i\, \bar\sigma\, \exp(h_{i,t}) \quad \text{and} \quad h_{i,t+1} = \rho_i\, h_{i,t} + s_i\, \eta_{i,t+1} \quad \text{for } i \in \{z, c, z\pi\}.$$

Here, z_t denote changes in the expected growth rate of consumption and $z_{\pi,t}$ is the expected inflation rate, which affects the mean growth rate of consumption through z_t. The log growth rate of $\{\lambda_t\}$ is independent of consumption and obeys

$$\ln\left(\frac{\lambda_{t+1}}{\lambda_t}\right) = h_{\lambda,t+1} = \rho_\lambda h_{\lambda,t} + s_\lambda\, \eta_{\lambda,t+1}. \tag{18}$$

All shocks are IID and standard normal. The state vector x is given by

$$x = (z, z_\pi, h_z, h_c, h_{z\pi}, h_\lambda) \in \mathsf{X} := \mathbb{R}^6.$$

[13] In terms of utility levels, an increase in the wealth-consumption ratio lowers utility when $\theta \geqslant 0$. As shown in Pohl et al. (2023), the existence problem shows up at the lower end in this case.

5.1. **Computing the Stability Coefficients.** In the framework stated above, independence of $\{\lambda_t\}$ and $\{C_t\}$ allows us to adopt the decomposition in Proposition 4.1 and derive an analytical expression for $\mathscr{S}_\lambda$, through which we sharpen and clarify the results in Proposition 4.2:[14]

Proposition 5.1. *If $\{\lambda_t\}$ obeys (18), then*

$$\mathscr{S}_\lambda = \theta \, \frac{s_\lambda^2}{2(1-\rho_\lambda)^2}. \tag{19}$$

Proof. By (18) we have $\lambda_T/\lambda_0 = \exp\left(\sum_{t=1}^T h_{\lambda,t}\right)$, and hence

$$\mathscr{S}_\lambda = \lim_{T\to\infty} \frac{1}{T} \ln \mathcal{R}_\theta \left(\exp\left(\sum_{t=1}^T h_{\lambda,t} \right)\right)$$

by (16). Note that

$$\sum_{t=1}^T h_{\lambda,t} \sim N(0, v_T) \quad \text{with} \quad v_T := \frac{s_\lambda^2}{(1-\rho_\lambda)^2}\left(T - \frac{2\rho_\lambda(1-\rho_\lambda^T)}{1-\rho_\lambda} + \frac{\rho_\lambda^2(1-\rho_\lambda^{2T})}{1-\rho_\lambda^2} \right).$$

It follows that

$$\ln \mathcal{R}_\theta \left(\exp\left(\sum_{t=1}^T h_{\lambda,t} \right)\right) = \frac{\theta v_T}{2}.$$

Multiplying by $1/T$ and taking the limit in T gives (19). □

Proposition 5.1 provides the strict inequality implication $\mathscr{S}_\lambda < 0$ whenever $\theta < 0$ and a direct connection between $\mathscr{S}_\lambda$ and the parameters in (18). Assuming $\gamma > 1$, as is standard in asset pricing models, $\psi > 1$ implies increases in volatility s_λ and persistence ρ_λ of the time preference shocks both decrease $\mathscr{S}_\lambda$, which makes the existence of a solution more likely. In contrast, if $\psi < 1$, then the sign of θ is reversed and the opposite is true.

Figure 2 plots $\mathscr{S}_\lambda$ as a function of ψ in the neighborhood of unity for the calibration used in Albuquerque et al. (2016).[15] As ψ approaches one, $|\mathscr{S}_\lambda|$ becomes large and significant: $\mathscr{S}_\lambda \to -\infty$ as $\psi \downarrow 1$ and $\mathscr{S}_\lambda \to \infty$ as $\psi \uparrow 1$. This is consistent with the strong effects of time preference shocks on equilibrium outcomes for ψ close to one reported in de Groot et al. (2018) and Kruger (2021). Given the decomposition (15), the stability coefficient $\mathscr{S}$ is dominated by the influence of time preference shocks as $\psi \to 1$. In other words, if ψ is close to but smaller than 1, $\mathscr{S}_\lambda$ is positive and large, making the existence of a solution

[14]Similar derivations to compute closed-form solutions for affine asset pricing models with CRRA utility can be found in Burnside (1998) and de Groot (2015).

[15]The qualitative conclusions we draw below are the same if we use the calibrations of Schorfheide et al. (2018) and Gomez-Cram and Yaron (2020).

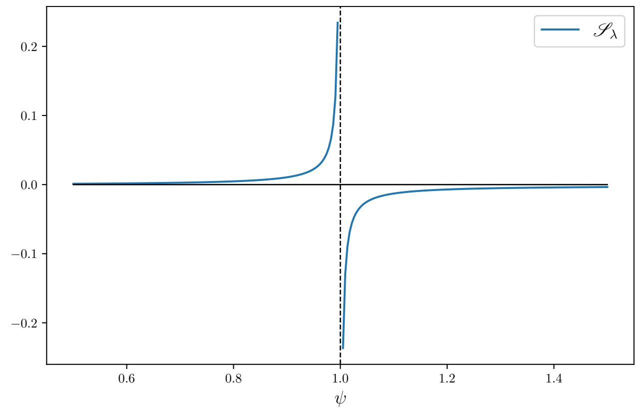

FIGURE 2. The figure plots $\mathscr{S}_\lambda$ as a function of ψ in the neighborhood of unity for the benchmark model of Albuquerque et al. (2016).

significantly less likely. In contrast, if ψ is close to but larger than 1, $\mathscr{S}_\lambda$ is negative and large such that even quickly growing economies can have a solution.

Next we analyze the quantitative effects of time preference shocks on existence for the different calibrations used in the three models. For the benchmark model of Albuquerque et al. (2016) (see Table 2), log consumption growth follows

$$g_{c,t+1} = \mu_c + \varphi_c \, \xi_{c,t+1}. \tag{20}$$

In this case we can derive an analytical result:[16]

Proposition 5.2. *Under the dynamics in* (18) *and* (20)*, we have*

$$\mathscr{S}_\lambda = \theta \, \frac{s_\lambda^2}{2(1-\rho_\lambda)^2} \quad \textit{and} \quad \mathscr{S}_c = \mu_c + \frac{1}{2}(1-\gamma)\varphi_c^2, \tag{21}$$

which implies that

$$\mathscr{S} = \ln \beta + \theta \, \frac{s_\lambda^2}{2(1-\rho_\lambda)^2} + \left(1 - \frac{1}{\psi}\right) \left[\mu_c + \frac{1}{2}(1-\gamma)\varphi_c^2\right]. \tag{22}$$

Proposition 5.2 shows that the effect of consumption risk also depends on ψ. For $\psi > 1$ the substitution effect dominates the wealth effect so the investor lowers her consumption relative to wealth in response to improved investment opportunities. So for a given $\mu_c > 0$, either β must be sufficiently small, φ_c must be sufficiently large or $h_{\lambda,t+1}$ must be sufficiently persistent and volatile to guarantee existence.

The models of Schorfheide et al. (2018) and Gomez-Cram and Yaron (2020) do not permit an analytical solution for $\mathscr{S}_c$. Hence, we compute the stability coefficient using

[16]The analytical expression for $\mathscr{S}_c$ in (21) is derived in Borovička and Stachurski (2020).

TABLE 1. Decomposition of the Stability Coefficient $\mathscr{S}$

Decomposition	Albuquerque et al. (2016)	SSY (2018)	GCY (2020)
$\mathscr{S}$	-0.0053	-0.00115	-0.0025
$\ln \beta$	-0.00205	-0.0010	-0.0013
$\mathscr{S}_\lambda$	-0.00375	-0.00076	-0.0016
$\left(1 - \frac{1}{\psi}\right) \mathscr{S}_c$	0.00049	0.00061	0.0004

Monte Carlo simulations (see Appendix A.4 for details). Table 1 lists the decomposition of $\mathscr{S}$ for the three models.

From Theorem 3.1, it follows that all three models have a unique stationary Markov solution in $\mathscr{C}$.[17] We also find that in all three specifications, the effect of the time preference shock on $\mathscr{S}$ is large: for any $\beta < 1$ we have $\mathscr{S} < 0$.

5.2. **Parameter Conditions for Existence.** Next we analyze how changes in other model parameters affect the existence of a solution, such as persistence and volatility of the state processes and preference parameters.

First, stronger discounting through β lowers the wealth-consumption ratio and hence makes the existence of a solution more likely. Annual calibrations in the literature usually consider values for β between 0.97 and 0.99. This implies monthly values between 0.9975 and 0.9992 and hence $\ln \beta \in [-0.0025, -0.0008]$. This gives a range for the maximal values of $\mathscr{S}_\lambda$ and $\mathscr{S}_c$ for which a solution can still exist. Furthermore, Theorem 3.1 implies that a sufficiently small β will always ensure the existence of a solution.

The effects of other model parameters on existence are more difficult to isolate, so we rely on numerical methods. In order to quantify the effect of a parameter, we increase or decrease its value while holding other parameters constant, until the stability coefficient turns positive. This gives a threshold value for the parameter that leads to nonexistence. By comparing the original value and the threshold value, we can quantify how close the parameter is to the boundary between stability and instability.

Table 2 lists for each model the threshold values of different parameters. For each of the models, we consider two cases: the original calibration with $\psi > 1$ and the case in which we set the IES to $1/\psi$. The latter case fundamentally changes the effects of the specific model parameters on existence as we have highlighted above.

[17]The only caveat is as follows: while these models have well-defined solutions at estimated parameters, this is not necessarily the case for all parameterizations used in their priors.

Table 2. Threshold Parameter Values for Non-Existence

Parameters		Albuquerque et al. (2016)			SSY (2018)			GCY (2020)		
		Original	Threshold		Original	Threshold		Original	Threshold	
			$\psi = 1.52$	$\psi = 0.66$		$\psi = 1.97$	$\psi = 0.51$		$\psi = 1.5$	$\psi = 0.67$
Preference	β	0.998	**1.003**	**0.998**	0.999	**1.000**	**1.001**	0.999	**1.001**	**1.000**
	γ	1.516	**0.786**	**1.553**	8.890	*	**15.03**	13.010	*	**22.04**
	ψ	1.457	**0.698**	**0.698**	1.970	**0.780**	**0.780**	1.500	**0.760**	**0.760**
Preference Shocks	ρ_λ	0.991	*	**0.992**	0.959	*	**0.983**	0.981	*	**0.986**
	s_λ	0.001	*	**0.001**	0.000	*	**0.001**	0.000	*	**0.000**
Consumption	μ_c	0.002	**0.019**	**0.001**	0.002	**0.004**	**-0.000**	0.002	**0.009**	**-0.000**
	ρ	-	-	-	0.987	*	**0.998**	0.983	*	**0.994**
Volatility	φ_z	-	-	-	0.000	*	**0.001**	0.000	*	**0.001**
	φ_c	0.007	*	**0.040**	0.004	*	**0.006**	0.002	*	**0.003**
	ρ_z	-	-	-	0.992	*	**0.999**	0.980	*	**0.997**
	ρ_c	-	-	-	0.991	*	**0.993**	0.992	*	**0.995**
	s_z	-	-	-	0.062	*	**0.163**	0.090	*	**0.210**
	s_c	-	-	-	0.098	*	**0.121**	0.104	*	**0.127**
Inflation	ρ_π	-	-	-	-	-	-	-0.007	*	**0.029**
	$\rho_{\pi\pi}$	-	-	-	-	-	-	0.985	*	**0.996**
	$\varphi_{z\pi}$	-	-	-	-	-	-	0.000	*	**0.000**
	$\rho_{z\pi}$	-	-	-	-	-	-	0.970	*	**0.983**
	$s_{z\pi}$	-	-	-	-	-	-	0.271	*	**0.351**

Notes: the table lists the threshold value for each model parameter that leads to nonexistence for the baseline model of Albuquerque et al. (2016), the model of Schorfheide et al. (2018), as well as the model of Gomez-Cram and Yaron (2020), assuming other parameters do not change. For each model, the three columns list the original parameter values, the threshold values under the original calibration, and the threshold values when the IES is changed to $1/\psi$. An "*" indicates that changing the parameter will not lead to nonexistence and a dash (-) means that the parameter is not applicable (and hence set to zero).

We first analyze the case of $\psi > 1$ (this corresponds to the second column under each model) as considered in the papers as well as most recent long-run risk asset pricing models. The first insight is that the only parameters that can lead to nonexistence are β, γ and μ_c. Increasing β or μ_c as well as decreasing γ will increase the wealth-consumption ratio and hence can lead to nonexistence.[18] For example, in the model of Schorfheide et al. (2018), an

[18] In the last two models, the threshold values for γ are negative, so we omit them in the table.

increase in μ_c from 0.0016 to 0.004 (from 0.0192 to 0.048 in annualized terms) can lead to nonexistence. All the other parameters add new sources of risks to the respective models. Our numerical exercise shows that their roles are similar to ρ_λ and s_λ in Proposition 5.1. When $\psi > 1$, increasing them makes the existence conditions less stringent.

On the other hand, if $\psi < 1$ (third column under each model), we obtain the opposite results. In this case, the income effect dominates the wealth effect so increasing risks in the economy and increasing risk aversion leads to a higher wealth-consumption ratio and hence can lead to nonexistence. For example, increasing persistence or volatility of any source of risk can change each model from having a well-defined solution to one where no solution exists.

6. An Alternative Specification

The way time preference shocks are introduced into Epstein–Zin utility in (1) implies extreme preference for early resolution of uncertainty and arbitrarily large responses to preference shocks as the IES approaches one (Kruger, 2021; de Groot et al., 2018). This specification also lacks several desirable properties of the standard Epstein–Zin preferences (e.g., γ may no longer represent the coefficient of relative risk aversion (de Groot et al., 2022)). To remedy these issues, de Groot et al. (2018) propose the following alternative specification for recursive preferences

$$V_t = \left[(1 - a_t\beta)C_t^{1-1/\psi} + a_t\beta \left\{\mathcal{R}_{t,1-\gamma}\left(V_{t+1}\right)\right\}^{1-1/\psi}\right]^{1/(1-1/\psi)}. \tag{23}$$

Comparing with (1), the time preference shock a_t in (23) appear before both current consumption and the continuation value, and their distributional weights sum to one.[19] de Groot et al. (2018) and de Groot et al. (2022) show that this specification does not suffer from the issues mentioned above.

To analyze the alternative specification, we divide both sides of (23) by C_t and raise it to the power of $1 - \gamma$, yielding

$$G'_t = \left\{(1 - a_t\beta) + a_t\beta \left[\mathbb{E}_t G'_{t+1} \left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]^{1/\theta}\right\}^{\theta}, \tag{24}$$

where $G'_t := (V_t/C_t)^{1-\gamma}$. As in Section 3, we seek a stationary Markov solution to (24) in the form of $\{G'_t\} := \{g(X_t)\}$. Assume $a_t = h(X_t) \in (0, 1/\beta)$, so that utility is always

[19] Kruger (2021) also considers an alternative specification where the time preference shock is a multiplier on C_t directly instead of on the flow utility $C_t^{1-1/\psi}$. In this section, we focus on the one from de Groot et al. (2018) because a_t in (23) has a similar role to λ_t in (1), as discussed below.

well-defined, and let $\tilde{\mathbb{K}}$ be defined by

$$(\tilde{\mathbb{K}}g)(x) = h^\theta(x)\mathbb{E}_x g(X_{t+1}) \exp\left\{(1-\gamma)g_c(X_t, X_{t+1}, \xi_{t+1})\right\}. \tag{25}$$

Then, the solution g satisfies

$$g(x) = (\tilde{\mathbb{T}}g)(x) := \left(1 - h(x)\beta + \left[\beta^\theta(\tilde{\mathbb{K}}g)(x)\right]^{1/\theta}\right)^\theta. \tag{26}$$

Similar to (10), we define

$$\mathscr{S}' := \ln\beta + \frac{\ln r(\tilde{\mathbb{K}})}{\theta}. \tag{27}$$

We obtain the following proposition, which parallels Theorem 3.1.

Proposition 6.1. *Let Assumption 2.1 hold. If $h \in \mathscr{C}$ and $\sup_{x\in\mathsf{X}} h(x) < 1/\beta$, then the following statements are equivalent:*

(a) $\mathscr{S}' < 0$.

(b) $\tilde{\mathbb{T}}$ *is globally geometrically stable on* $\mathscr{C}$.

Moreover, if $\mathscr{S}' \geqslant 0$, then no stationary Markov solution to (24) exists in $\mathscr{C}$.

Comparing the proposition above and Theorem 3.1, the difference lies in the definition of the discount operator. If we write $h^\theta(x)$ in (25) as $\exp\{\theta \ln h(x)\}$, it becomes clear that the variable a_t here has the same role as λ_{t+1}/λ_t in (3) under different timing. But now a_t cannot exceed $1/\beta$ for the utility to be well-defined, which provides an upper bound on the growth rate of time preference shocks. This is a major source of the different model characteristics under the alternative specification.

To gain some further insight, we let

$$(\mathbb{K}_c g)(x) := \mathbb{E}_x g(X_{t+1}) \exp\left\{(1-\gamma)g_c(X_t, X_{t+1}, \xi_{t+1})\right\} \tag{28}$$

be a discount operator that only takes into account the role of consumption growth and set $\bar{a} := \sup_{x\in\mathsf{X}} h(x)$. Then we have the following corollary to Proposition 6.1.

Corollary 6.2. *Under the assumptions of Proposition 6.1, if*

$$\mathscr{S}'' := \ln\beta + \ln\bar{a} + \frac{\ln r(\mathbb{K}_c)}{\theta} < 0, \tag{29}$$

then $\tilde{\mathbb{T}}$ is globally geometrically stable on $\mathscr{C}$.

Corollary 6.2 gives a sufficient condition for existence and uniqueness of recursive utility. In particular, (29) does not contain the growth rate of time preference shocks. Instead, only the maximum level of $\{a_t\}$ affects existence. In this case, no matter whether ψ approaches 1 from above or below, $\lim_{\psi\to 1}\mathscr{S}'' = \ln\beta + \ln\bar{a}$. This confirms the lack of asymptote reported in de Groot et al. (2018) and de Groot et al. (2022).

Unit IES. de Groot et al. (2018) emphasize that the alternate specification (23) has a well-defined limit as ψ approaches unity, while the original specification (1) does not. Hence, as our final task, we consider the special case of a unit IES for the alternative specification.

As $\psi \to 1$, (23) becomes

$$V_t = C_t^{1-a_t\beta}\left[\mathcal{R}_{t,1-\gamma}\left(V_{t+1}\right)\right]^{a_t\beta} = C_t^{1-a_t\beta}\left[\mathbb{E}V_{t+1}^{1-\gamma}\right]^{a_t\beta/(1-\gamma)}. \tag{30}$$

Since (30) has a different functional form to the recursive utility defined in (1) or (23), it does not fit in our theoretical framework in Section 3. Intuitively, suppose that Proposition 6.1 extends to the case of $\psi = 1$. Since the stability coefficient $\mathscr{S}' \to \ln\beta < 0$ as $\psi \to 1$, there should be a unique Markov solution to (30) regardless of model parameters according to the proposition. We will show that this is indeed the case.

To see this, we again define $G'_t := (V_t/C_t)^{1-\gamma}$, so (30) can be written equivalently as

$$G'_t = \left[\mathbb{E}_t G'_{t+1}\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]^{a_t\beta}. \tag{31}$$

Then, a Markov solution g satisfies

$$g(x) = (\tilde{T}g)(x) := \left[(\mathbb{K}_c g)(x)\right]^{h(x)\beta}, \tag{32}$$

where $\mathbb{K}_c$ is as defined in (28) and $a_t = h(X_t)$ as above. We have the following proposition.

Proposition 6.3. *Let Assumption 2.1 hold. If $h \in \mathscr{C}$ and $\sup_{x\in\mathsf{X}} h(x) < 1/\beta$, then $\tilde{\mathbb{T}}$ is globally geometrically stable on $\mathscr{C}$ and there is a unique Markov solution to (30) in $\mathscr{C}$.*

Note that the case of standard Epstein–Zin preferences with $\psi = 1$ and no time preference shocks is nested in this framework by setting $a_t = h(X_t) \equiv 1$. As long as $\beta < 1$ and Assumption 2.1 is satisfied, there exists a unique solution according to Proposition 6.3.

7. Conclusion

This paper provides conditions to precisely characterize existence and uniqueness of recursive utility in a broad range of asset pricing models with time preference shocks. Our approach relies on a stability coefficient that allows for a clear interpretation in terms of preference parameters, preference shock dynamics, and the dynamics of consumption paths. We apply our theory to a class of long-run risk models and identify the admissible parameter region where a solution exists as well as the region where there is no solution. This provides practical guidelines for working with asset pricing models with time preference shocks.

As our approach does not require strong assumptions on the underlying consumption process, it can be extended to cases in which time preference shocks are added to models that rely on other risk factors such as consumption disasters, volatility of volatility, and jumps in volatility and growth rates. The details of these extensions and the impact of other risk factors on the stability coefficient are left for future research.

Appendix A. Appendix

In this section we detail computations and collect remaining proofs. As a first step, a general fixed point theorem from Stachurski et al. (2022) is provided below, along with some mathematical preliminaries.

A.1. **Preliminaries.** Let $\mathcal{E} := (\mathcal{E}, \|\cdot\|, \leqslant)$ be a Banach lattice (see, e.g, Meyer-Nieberg (2012)). For a linear operator A mapping $\mathcal{E}$ to itself, the operator norm and spectral radius of A are defined, as usual, by $\|A\| := \sup\{\|Ag\| : g \in \mathcal{E},\ \|g\| \leqslant 1\}$ and $r(A) := \lim_{n\to\infty} \|A^n\|^{1/n}$ respectively. The operator A is called *positive* if $Ag \geqslant 0$ whenever $g \geqslant 0$. It is called *bounded* if $\|A\|$ is finite and *compact* if the image of the unit ball in $\mathcal{E}$ under A has compact closure. A positive operator A is called *irreducible* if the only nontrivial ideal on which A is invariant is the whole space $\mathcal{E}$.[20]

For a self-map S from $E \subset \mathcal{E}$ to itself, we will say that S is geometrically stable on E if S has a unique fixed point u^* in E and, moreover, for all $u \in E$, there exists an $a < 1$ and $N < \infty$ such that $\|S^n u - u^*\| \leqslant a^n N$ for all $n \in \mathbb{N}$.

Next, we restrict attention to $C(\mathsf{X})$, the space of continuous functions on X. When paired with the supremum norm and the pointwise partial order, $C(\mathsf{X})$ is a Banach lattice. Let $T : C(\mathsf{X}) \to C(\mathsf{X})$ be defined by

$$Tf = \left((Af)^{1/s} + b\right)^s, \tag{33}$$

where A is a positive linear operator on $C(\mathsf{X})$, $b \in C(\mathsf{X})$ and $s \in \mathbb{R}$ with $s \neq 0$. Notice the similarity between the right-hand side of (33) and the outer function F in (6). In fact, it can be shown that the operator (9) is a special case of (33). Recall that $\mathscr{C}$ is the space of everywhere positive functions in $C(\mathsf{X})$. This corresponds to the interior of the positive cone of $C(\mathsf{X})$. We have the following theorem that characterizes existence and uniqueness of fixed points of (33).

Theorem A.1 (Stachurski et al. (2022))**.** *If $A\colon C(\mathsf{X}) \to C(\mathsf{X})$ is irreducible and eventually compact*[21] *and $b \in \mathscr{C}$, then the following statements are equivalent:*

[20] An ideal I in $\mathcal{E}$ is a vector subspace of $\mathcal{E}$ such that $g \in I$ and $f \in \mathcal{E}$ with $|f| \leqslant |g|$ implies $f \in I$.

[21] A linear operator A is eventually compact if there exists a $k \in \mathbb{N}$ such that A^k is compact.

(a) $r(A)^s < 1$.

(b) T *is geometrically stable on* $\mathscr{C}$.

Moreover, if $r(A)^s \geqslant 1$*, then* G *has no fixed point in* $\mathscr{C}$.

A.2. **Properties of The Discount Operator.** In this section we investigate the properties of $\mathbb{K}$. We adopt the setting of Section 2 and all assumptions stated there are maintained here, including Assumption 2.1.

Let $L_1(\pi)$ be all Borel measurable functions $g \colon \mathsf{X} \to \mathbb{R}$ with

$$\|g\| := \int |g(x)| \pi(\mathrm{d}x) < \infty.$$

For $f, g \in L_1(\pi)$ we write $f \leqslant g$ if $f(x) \leqslant g(x)$ for π-almost all $x \in \mathsf{X}$. We write $f \ll g$ if $f(x) < g(x)$ for π-almost all $x \in \mathsf{X}$. We define $\mathscr{G}$ to be all $f \in L_1(\pi)$ such that $f \gg 0$.

The next lemma will be useful.

Lemma A.2. *The common marginal density* π *of each* X_t *is continuous and everywhere positive on* $L_1(\pi)$.

Proof. This follows from Assumption 2.1. The proof is identical to that of Lemma C1 in Borovička and Stachurski (2020). □

We will use a kind of local spectral radius theorem. The version below is proved in Borovička and Stachurski (2020).

Theorem A.3. *Let* A *be a linear operator on* $L_1(\pi)$*. If* A *is eventually compact and* $Ag \in \mathscr{G}$ *whenever* $g \in \mathscr{G}$*, then*[22]

$$\lim_{n \to \infty} \left\{ \int A^n h \, \mathrm{d}\pi \right\}^{1/n} = r(A) \quad \textit{for all } h \in \mathscr{G}. \tag{34}$$

Below we use Theorem A.3 to generate the alternative representation of $\mathscr{S}$ provided in Proposition 4.1.

Throughout this section, we regard $\mathbb{K}$ from (7) as a linear operator on $L_1(\pi)$. The spectral radius $r(\mathbb{K})$ of $\mathbb{K}$ is the $L_1(\pi)$ spectral radius. We find it convenient to express $\mathbb{K}$ as the integral operator

$$(\mathbb{K}g)(x) = \int g(y) k(x, y) \, \mathrm{d}y \tag{35}$$

[22] In Theorem A.1, $r(A)$ is the spectral radius of A with respect to the supremum norm on $C(\mathsf{X})$, while here $r(A)$ is used to denote the $L_1(\pi)$ spectral radius, with a slight abuse of notation. Nonetheless, as we will show below, the two spectral radii are the same for $\mathbb{K}$.

where the kernel k is given by $k(x,y) := \mathbb{E}_\xi \Gamma(x,y,\xi) q(x,y)$ for $(x,y) \in \mathsf{X} \times \mathsf{X}$, with Γ defined in (8). Since $\mathbb{E}_\xi \Gamma$ and q are both continuous and X is compact, the function k is continuous and bounded on $\mathsf{X} \times \mathsf{X}$.

Lemma A.4. *Regarding the operator $\mathbb{K}$, the following statements are true:*

(a) $\mathbb{K}$ is a bounded linear operator on $L_1(\pi)$.

(b) $\mathbb{K}g$ is continuous for all $g \in L_1(\pi)$.

(c) $\mathbb{K}g \geqslant 0$ when $g \geqslant 0$ and $\mathbb{K}g \in \mathscr{G}$ whenever $g \in \mathscr{G}$.

(d) $\mathbb{K}$ is irreducible and $\mathbb{K}^2$ is compact.

Proof of Lemma A.4. Proofs for (a)–(b) can be found in the proof of Lemma C2 of Borovička and Stachurski (2020). (While the definition of the kernel k for the integral operator $\mathbb{K}$ is different in Borovička and Stachurski (2020), its properties are essentially identically. As a result, no modifications to the proof are necessary.) For part (c), the first claim is obvious and the second follows from everywhere positivity of Γ. Part (d) follows from Lemma C3 of Borovička and Stachurski (2020). $\square$

Lemma A.5. *For all $n \in \mathbb{N}$, we have*

$$(\mathbb{K}^n \mathbb{1})(x) = \mathbb{E}_x \left(\frac{\lambda_n}{\lambda_0}\right)^\theta \left(\frac{C_n}{C_0}\right)^{1-\gamma} \qquad (x \in \mathsf{X}). \tag{36}$$

Proof. Fix $n \in \mathbb{N}$. A straightforward inductive argument confirms that

$$(\mathbb{K}^n \mathbb{1})(x) = \mathbb{E}_x \prod_{i=1}^n \Gamma(X_{i-1}, X_i, \xi_i) \tag{37}$$

for all $x \in \mathsf{X}$, where $\mathbb{E}_x$ conditions on $X_0 = x$. Now observe that

$$\begin{aligned}
\prod_{i=1}^n \Gamma(X_{i-1}, X_i, \xi_i) &= \prod_{i=1}^n \exp\left\{\theta g_\lambda(X_{i-1}, X_i, \xi_i) + (1-\gamma) g_c(X_{i-1}, X_i, \xi_i)\right\} \\
&= \prod_{i=1}^n \left(\frac{\lambda_i}{\lambda_{i-1}}\right)^\theta \left(\frac{C_i}{C_{i-1}}\right)^{1-\gamma}.
\end{aligned}$$

Cancelling terms and combining with (37) gives (36). $\square$

Proof of Proposition 4.1. From Lemma A.5 and the law of iterated expectations we have

$$\int \mathbb{K}^n \mathbb{1} \, \mathrm{d}\pi = \mathbb{E} \left(\frac{\lambda_n}{\lambda_0}\right)^\theta \left(\frac{C_n}{C_0}\right)^{1-\gamma},$$

where $\mathbb{E}$ is the unconditional stationary expectation (i.e., with $X_0 \stackrel{\mathrm{d}}{=} \pi$). By independence of $\{\lambda_t\}$ and $\{C_t\}$, we then have

$$\begin{aligned}\left\{\int \mathbb{K}^n \mathbb{1} \, \mathrm{d}\pi\right\}^{1/n} &= \left\{\mathbb{E}\left(\frac{\lambda_n}{\lambda_0}\right)^{\theta}\right\}^{1/n} \left\{\mathbb{E}\left(\frac{C_n}{C_0}\right)^{1-\gamma}\right\}^{1/n} \\ &= \left\{\mathcal{R}_\theta\left(\frac{\lambda_n}{\lambda_0}\right)\right\}^{\theta/n} \left\{\mathcal{R}_{1-\gamma}\left(\frac{C_n}{C_0}\right)\right\}^{(1-\gamma)/n}.\end{aligned}$$

In view of Lemma A.4, the operator $\mathbb{K}$ satisfies all the conditions of the local spectral radius result in Theorem A.3. Hence, taking the limit and raising to the power of $1/\theta$, we have

$$r(\mathbb{K})^{1/\theta} = \lim_{n\to\infty} \left\{\mathcal{R}_\theta\left(\frac{\lambda_n}{\lambda_0}\right)\right\}^{1/n} \lim_{n\to\infty} \left\{\mathcal{R}_{1-\gamma}\left(\frac{C_n}{C_0}\right)\right\}^{(1-1/\psi)/n}.$$

Multiplying by β and taking logs yields

$$\mathscr{S} := \ln\beta + \frac{\ln r(\mathbb{K})}{\theta} = \ln\beta + \mathscr{S}_\lambda + \left(1 - \frac{1}{\psi}\right)\mathscr{S}_c,$$

as was to be shown. $\square$

Proof of Proposition 4.2. In view of (3), we have

$$\mathscr{S}_\lambda = \lim_{T\to\infty} \frac{1}{T} \ln \mathcal{R}_\theta\left(\frac{\lambda_T}{\lambda_0}\right) = \frac{1}{\theta} \lim_{T\to\infty} \frac{1}{T} \ln \mathbb{E}\exp\left(\theta \sum_{t=1}^{T} g_t\right),$$

where $g_t := g_\lambda(X_{t-1}, X_t, \xi_t)$. Since g_t has zero mean, so does $Z_T := \theta\sum_{t=1}^T g_t$. As a result, $\ln\mathbb{E}\exp(Z_T) \geqslant 0$.[23] It follows directly that $\mathscr{S}_\lambda \cdot \theta \geqslant 0$. $\square$

A.3. **Remaining Proofs.** We now complete all remaining proofs. As before, Assumption 2.1 is in force.

By (b) of Lemma A.4, we know that the discount operator $\mathbb{K}$ maps $L_1(\pi)$ into $C(\mathsf{X})$, so $\mathbb{K}$ is also a self-map on $C(\mathsf{X})$. In order to apply Theorem A.1, we need the following lemma.

Lemma A.6. *The operator* $\mathbb{K} \colon C(\mathsf{X}) \to C(\mathsf{X})$ *is irreducible and compact.*

Proof. Suppose that there exists a non-trivial ideal $I \subsetneq C(T)$ on which $\mathbb{K}$ is invariant. Then $I = \{f \in C(\mathsf{X}) : f(x) = 0, \forall x \in K\}$ for some closed nonempty K (see, e.g., Meyer-Nieberg, 2012, Proposition 2.1.9). By Assumption 2.1 and the positivity of Γ, there exists an $\ell \in \mathbb{N}$ such that $\mathbb{K}^\ell g \gg 0$ for some $g \in I$. This is a contradiction. Therefore, $\mathbb{K}$ is irreducible.

For compactness, we need to show that $\mathbb{K}$ maps the unit ball B_1 in $C(\mathsf{X})$ to a relatively compact set. Since X is compact, by the Arzelà–Ascoli theorem, it suffices to show that

[23] If Z satisfies $\mathbb{E}Z = 0$, then, by Jensen's inequality, $0 = \mathbb{E}\ln\exp(Z) \leqslant \ln\mathbb{E}\exp(Z)$.

$\mathbb{K}(B_1)$ is bounded and equicontinuous. Since $k(x, y)$ is continuous and X is compact, $\|\mathbb{K}g\| \leqslant \sup_x \int k(x, y)\, \mathrm{d}y = M < \infty$ for all $g \in B_1$. By (b) of Lemma A.4, $\mathbb{K}g$ is continuous for all $g \in B_1$. Hence, $\mathbb{K}(B_1)$ is relatively compact and $\mathbb{K}$ is an compact operator on $C(\mathsf{X})$. $\square$

The next lemma shows the equivalence of the spectral radius of $\mathbb{K}$ on $L_1(\pi)$ and $C(\mathsf{X})$. This allows us to connect the general fixed point Theorem A.1 to the stability coefficient $\mathscr{S}$. Let $r(A; \mathcal{E})$ denote the spectral radius of a linear operator A on a Banach space $\mathcal{E}$.

Lemma A.7. *$r(\mathbb{K}; L_1(\pi)) = r(\mathbb{K}; C(\mathsf{X}))$.*

Proof. Since $\mathbb{K}$ is irreducible and compact on $C(\mathsf{X})$, there exists an $e \in C(\mathsf{X})$ with $e \gg 0$ such that

$$\mathbb{K}e = r(\mathbb{K}; C(\mathsf{X}))e \tag{38}$$

and $r(\mathbb{K}; C(\mathsf{X})) > 0$ (Meyer-Nieberg, 2012, Theorem 4.1.4 and Lemma 4.2.9). Since X is compact, $e \in L_1(\pi)$. By Theorem A.3,

$$r(\mathbb{K}; L_1(\pi)) = \lim_{n\to\infty} \left\{ \int \mathbb{K}^n e \, \mathrm{d}\pi \right\}^{1/n} = \lim_{n\to\infty} \left\{ \int r^n(\mathbb{K}; C(\mathsf{X}))e \, \mathrm{d}\pi \right\}^{1/n} = r(\mathbb{K}; C(\mathsf{X})),$$

where the second equality follows from iterating on (38). $\square$

Proof of Theorem 3.1. Note that the operator $\mathbb{T}$ as defined in (9) can be written as

$$(\mathbb{T}g)(x) = \left(1 - \beta + [\beta^\theta (\mathbb{K}g)(x)]^{1/\theta}\right)^\theta,$$

which is a special case of (33) with $b \equiv 1 - \beta$, $s = \theta$, and $Ag = \beta^\theta \mathbb{K}g$. By Lemma A.6 and Theorem A.1, it suffices to show that $r(A)^\theta < 1$ if and only if $\mathscr{S} < 0$. Since $r(A) > 0$ (Meyer-Nieberg, 2012, Lemma 4.2.11), $r(A)^\theta < 1$ if and only if $\ln r(A)$ and θ have opposite signs. Then we have

$$r(A)^\theta < 1 \iff \frac{\ln r(A)}{\theta} < 0 \iff \ln \beta + \frac{\ln r(\mathbb{K})}{\theta}.$$

In view of the definition of $\mathscr{S}$ in (10) and Lemma A.7, the proof is now complete. $\square$

Proof of Proposition 3.2. Let $\tau \colon \mathscr{C} \to \mathscr{C}$ be defined by $\tau g = (1 - \beta)^{-1} g^{1/\theta}$. Straightforward algebra shows that $\mathbb{U} = \tau \mathbb{T} \tau^{-1}$ on $\mathscr{C}$. Since $\theta \neq 0$ and $\beta \in (0, 1)$, the map τ and its inverse $\tau^{-1} f = (1 - \beta)^\theta f^\theta$ are continuous on $\mathscr{C}$ when $\mathscr{C}$ is endowed with the supremum norm distance. As a result, $(\mathscr{C}, \mathbb{T})$ and $(\mathscr{C}, \mathbb{T})$ are topologically conjugate dynamical systems, which in turn implies that $(\mathscr{C}, \mathbb{U})$ is globally stable if and only if $(\mathscr{C}, \mathbb{T})$ is globally stable. The claim in Proposition 3.2 now follows from Theorem 3.1. $\square$

Proof of Proposition 6.1. Similar to the proof of Theorem 3.1, we aim to apply Theorem A.1.

First note that all statements of Lemmas A.4, A.6, and A.7 also hold for the operator $\tilde{\mathbb{K}}$ because $h \in \mathscr{C}$ and X is compact. Moreover, (26) is a special case of (33) with $b(x) = 1 - h(x)\beta$, $s = \theta$, and $Ag = \beta^\theta \tilde{\mathbb{K}} g$. Since $\sup_{x \in \mathsf{X}} h(x) < 1/\beta$ and $h \in \mathscr{C}$, $b \in \mathscr{C}$ and the assumptions of Theorem A.1 are all satisfied. The theorem then follows from the fact that $r(A)^\theta < 1$ is equivalent to $\mathscr{S}' < 0$. □

Proof of Corollary 6.2. Since $\bar{a} = \sup_{x \in \mathsf{X}} h(x)$, $r(\tilde{\mathbb{K}}) \leqslant r(\bar{a}^\theta \mathbb{K}_c)$. It follows that $\mathscr{S}'' > \mathscr{S}'$ and $\mathscr{S}'' < 0$ is a sufficient condition of $\mathscr{S}' < 0$. The conclusion then follows directly from Proposition 6.1. □

Proof of Proposition 6.3. We shall apply Theorem 2.2 in Stachurski et al. (2022). Since $0 < h(x)\beta < 1$ for all $x \in \mathsf{X}$, the operator $\tilde{T}$ is order-preserving and concave on $\mathscr{C}$. Similar to Lemma A.4, we can show that $\tilde{T}$ is a self-map on $\mathscr{C}$. In order to apply Theorem 2.2 in Stachurski et al. (2022), it remains to prove that

(a) for all $f \in \mathscr{C}$, there exists a $p \in \mathscr{C}$ such that $p \leqslant f$ and $\tilde{T} p \gg p$, and
(b) for all $f \in \mathscr{C}$, there exists $q \in \mathscr{C}$ such that $f \leqslant q$ and $\tilde{T} q \leqslant q$.

Similar to the proof of Lemma A.6, we can show that $\mathbb{K}_c$ is compact and irreducible, so there exists an $e \in \mathscr{C}$ such that $\mathbb{K}_c e = re$ with $r > 0$. To prove the above two claims, fix $f \in \mathscr{C}$ and consider $g = ce$ with $c > 0$. We have

$$(\tilde{T} g)(x) = [\mathbb{K}_c g(x)]^{h(x)\beta} = [rce(x)]^{h(x)\beta} \quad \text{and} \quad \frac{(\tilde{T} g)(x)}{g(x)} = r^{h(x)\beta} \, [ce(x)]^{h(x)\beta - 1}.$$

Since $h(x)\beta - 1 < 0$, $c \to 0$ implies that $\tilde{T} g(x)/g(x) \to \infty$ for all x. Since X is compact, there exists an $\epsilon_1 > 0$ such that $\tilde{T} g \gg g$ for all $c \leqslant \epsilon_1$. For the same reason, there exists an $\epsilon_2 > 0$ such that $g \leqslant f$ for all $c \leqslant \epsilon_2$. Setting $p = c_1 e$ with $c_1 = \min\{\epsilon_1, \epsilon_2\}$ proves the first claim. Similarly, $c \to \infty$ implies that $\tilde{T} g / g \to 0$ uniformly, so we can find a $c_2 > 0$ such that $f \leqslant q$ and $\tilde{T} q \leqslant q$ where $q = c_2 e$. By Theorem 2.2 of Stachurski et al. (2022), $\tilde{T}$ is globally geometrically stable on $\mathscr{C}$. As a result, there is a unique Markov solution in $\mathscr{C}$. □

A.4. **Numerical Accuracy.** In this section, we compute the stability coefficient $\mathscr{S}$ using Monte Carlo simulations for models that do not have an analytic expression for $\mathscr{S}_c$. Recall that $\mathscr{S}_c$ is given by

$$\mathscr{S}_c = \lim_{T \to \infty} \frac{1}{T} \ln \mathcal{R}_{1-\gamma} \left(\frac{C_T}{C_0} \right) = \lim_{T \to \infty} \frac{1}{T} \frac{1}{1-\gamma} \ln \mathbb{E} \left(\frac{C_T}{C_0} \right)^{1-\gamma}$$

by Proposition 4.1. To estimate it numerically, we generate N independent consumption paths of length T and evaluate

$$\hat{\mathscr{S}}_c = \frac{1}{T}\frac{1}{1-\gamma}\ln\frac{1}{N}\sum_{n=1}^{N}\left(\frac{C_T^{(n)}}{C_0^{(n)}}\right)^{1-\gamma},$$

where $C_t^{(n)}$ is the nth simulation of time t consumption. Here, we use the sample average to estimate the expectation in $\mathscr{S}_c$, the validity of which is guaranteed by the Law of Large Numbers.

(A) Schorfheide et al. (2018) (B) Gomez-Cram and Yaron (2020)

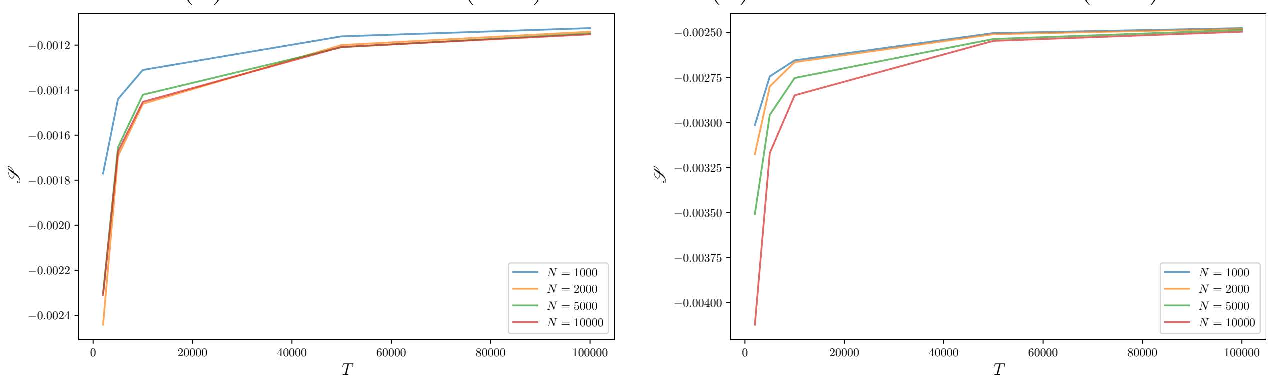

FIGURE 3. The figure plots $\mathscr{S}$ for different N and T for the models of Schorfheide et al. (2018) and Gomez-Cram and Yaron (2020).

To evaluate how sample size and length affect the accuracy of the estimator $\hat{\mathscr{S}}_c$, we plot the estimated stability coefficient for different N and T for the models of Schorfheide et al. (2018) and Gomez-Cram and Yaron (2020) in Figure 3. The graphs show that for large N and T, increasing them further only has a marginal effect on the estimates. For the results reported in the main text, we use $T = 100,000$ and $N = 10,000$.[24]

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