Coase Meets Bellman: Dynamic Programming for Production Networks

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Outline

Problem Description

$$v^*(\hat{x}) := \min_{\{a_t\}} \sum_{t=0}^{\infty} \beta^t \ell(a_t)$$

$$\text{s.t.} \sum_{t=0}^{\infty} a_t = \hat{x}, \ a_t \geq 0$$

- The agent takes action a_t in period t and suffers loss $\ell(a_t)$
- Assume that $\beta > 1$, $\ell(0) = 0$, $\ell' > 0$, and $\ell'' > 0$
- It doesn't help to transform it to a maximization problem

$$\max_{\{a_t\}} \sum_{t=0}^{\infty} \beta^t (-\ell(a_t))$$

- Lagrange multipliers?

Problem Description

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$$\text{s.t. } \sum_{t=0}^{\infty} a_t = \hat{x}, \; a_t \geq 0$$

- Set $x_0 = \hat{x}$ and $x_{t+1} = x_t - a_t$
- Does v^* satisfy the Bellman equation?

$$v(x) = \min_{0 \leq a \leq x} \{\ell(a) + \beta v(x-a)\}$$

- Why study the Bellman equation?
  1. Use results from dynamic programming
  2. Different economic interpretations

Outline

An Abstract Dynamic Program

- A dynamic program consists of
  1. State space X
  2. Action space A
  3. Feasible correspondence $\Gamma(x) \subset A$
  4. Continuation aggregator $H(x, a, v)$
- The Bellman equation is

$$v(x) = (Tv)(x) := \inf_{a \in \Gamma(x)} H(x, a, v)$$

- Example: $X = A = [0, \hat{x}]$, $\Gamma(x) = [0, x]$, and

$$H(x, a, v) = \ell(a) + \beta v(x - a)$$

Fixed Point Results

Assumptions:

1. $H(x, a, v)$ is increasing and concave in v
2. There exist ϕ, ψ such that T is a self-map on $[\phi, \psi]$
3. Other technical conditions

Theorem

T has a unique fixed point $\bar{v} \in [\phi, \psi]$, $T^n v \to \bar{v}$ for all $v \in [\phi, \psi]$, and $\bar{\sigma}(x) := \arg\min_{a \in \Gamma(x)} H(x, a, \bar{v})$ is upper hemicontinuous. Under additional assumptions, $\bar{v}$ is increasing, convex, and continuously differentiable on $\operatorname{int} X$ and $\bar{v}'(x) = H_x(x, \bar{\sigma}(x), \bar{v})$.

Optimality

Definitions

- Any σ corresponds to a policy operator

$$(T_\sigma v)(x) := H(x, \sigma(x), v)$$

- For a sequence of feasible policies $\pi = (\sigma_0, \sigma_1, \ldots)$, define the π-value function

$$v_\pi(x) = \lim_{n \to \infty} (T_{\sigma_0} T_{\sigma_1} \ldots T_{\sigma_n} \phi)(x)$$

- The value function is $v^*(x) := \inf_\pi v_\pi(x)$
- The optimal policy π^* is such that $v^* = v_{\pi^*}$
- Stationary policies $(\sigma, \sigma, \ldots)$

Optimality

Assumptions:

1. If $v_n \uparrow v$, $H(x, a, v_n) \to H(x, a, v)$
2. There exists $\beta > 0$ such that for all $r > 0$

$$H(x, a, v + r) \leq H(x, a, v) + \beta r$$

Theorem

The value function solves the Bellman equation and $v^* = \bar{v}$. There exists a stationary optimal policy σ^* and $\sigma^* = \bar{\sigma}$.

Outline

Negative Discount Dynamic Programming

Recall that $x_{t+1} = x_t - a_t$ and $x_0 = \hat{x}$. We aim to find σ^* such that $a_t^* = \sigma^*(x_t)$ solves

$$v^*(\hat{x}) := \min_{\{a_t\}} \sum_{t=0}^{\infty} \beta^t \ell(a_t) \quad \text{s.t. } \sum_{t=0}^{\infty} a_t = \hat{x}, \; a_t \geq 0$$

Let $\phi(x) = \ell'(0)x$ and $\psi(x) = \ell(x)$

Theorem

1. v^* is a fixed point of the Bellman operator

$$(Tv)(x) := \inf_{0 \leq a \leq x} \{\ell(a) + \beta v(x - a)\}$$

and $T^n v \to v^*$ for all $v \in [\phi, \psi]$

Negative Discount Dynamic Programming

Theorem (cont.)

2. There is a unique optimal policy given by

$$\sigma^*(x) = \operatorname*{arg\,min}_{0 \leq a \leq x} \{\ell(a) + \beta v^*(x-a)\},$$

   and σ^* is increasing

3. v^* is strictly increasing, strictly convex, and continuously differentiable on $(0, \hat{x})$ with $(v^*)'(x) = \ell'(\sigma^*(x))$

4. $\{a_t^*\}$ is decreasing and satisfies

$$\ell'(a_{t+1}^*) = \max\left\{\frac{1}{\beta}\ell'(a_t^*), \ell'(0)\right\}$$

Negative Discount Dynamic Programming

Theorem

When $\ell'(0) = 0$, $\{a_t^*\}$ is optimal if and only if $\beta \ell'(a_{t+1}^*) = \ell'(a_t^*)$. The sequence is unique, decreasing, and strictly positive. Furthermore, $v^* = Tv^*$.

To solve the Bellman equation:

- If $\ell'(0) > 0$, we use the Bellman operator directly
- If $\ell'(0) = 0$, we use the Euler equation

Outline

Reinterpretation

- Time index → an index over decision making entities
- Tasks are completed sequentially

$$\text{total cost for agent 0} = \ell(a_0) + \beta\Big[\overbrace{\ell(a_1) + \beta\big(\underbrace{\ell(a_2) + \beta(\dots)}_{\text{total cost for agent 2}}\big)}^{\text{total cost for agent 1}}\Big]$$

- A social planner's problem

$$v^*(\hat{x}) := \min_{\{a_t\}} \sum_{t=0}^{\infty} \beta^t \ell(a_t) \quad \text{s.t.} \ \sum_{t=0}^{\infty} a_t = \hat{x}, \ a_t \geq 0$$

- A decentralized problem

$$v^*(x) := \min_{0 \leq a \leq x} \{\ell(a) + \beta v^*(x - a)\}$$

Production Chains

Kikuchi et al. (2018)

- Competitive market with price-taking firms indexed by i
- Produce a unit of good: implementing a sequence of tasks
- The allocation of tasks $\{a_i\}$ satisfies $\sum_i a_i = 1$
- Define firm boundaries: $b_0 = 1$ and $b_{i+1} = b_i - a_i$
- Profits of the ith firm are

$$\pi_i = p(b_i) - c(a_i) - (1 + \tau)p(b_{i+1})$$

where p is price, c is production cost, and τ is transaction cost

Production Chains

Kikuchi et al. (2018)

Definition

The pair $(p, \{a_i\})$ is called an *equilibrium* for the production chain if

1. $\pi_i = 0$ for all i.
2. $p(s) - c(s - t) - (1 + \tau)p(t) \leq 0$ for any pair s, t with $0 \leq t \leq s \leq 1$, and

Theorem

Let $\hat{x} = 1$, $\ell = c$, and $\beta = 1 + \tau$. Then, $(v^*, \{a_i^*\})$ is an equilibrium for the production chain.

City Hierarchy

- An alternative theory of city size distribution (Hsu, 2012; Hsu et al., 2014)
- Cities are built to host a continuum of dwellers of measure one
- A government opens competition for city developers
- Each developer builds a city of a certain size and pays other developers to build "satellite cities"
- Layers of cities are formed indexed by i
- Price function p, building cost c, tax rate τ

City Hierarchy

- A developer assigned to host s dwellers maximizes profits

$$\max_{0 \leq t \leq s} \{p(s) - c(s-t) - (1+\tau)kp(t/k)\}$$

- The equilibrium price function satisfies

$$p(s) = \min_{0 \leq t \leq s} \{c(s-t) + (1+\tau)kp(t/k)\}$$

- Let $c(s) = s^\gamma$ with $\gamma > 1$ and let $k = 2$

City Hierarchy

- The Euler equation: $c'(a_i^*) = (1+\tau)c'(a_{i+1}^*)$
- If $\theta := (1+\tau)^{1/(1-\gamma)} < 1/2$, then $a_i^* = \theta^i(1-2\theta)$ and $v^*(s) = (1-2\theta)^{\gamma-1}s^\gamma$
- $p = v^*$ solves the Bellman equation and is an equilibrium price

City Hierarchy

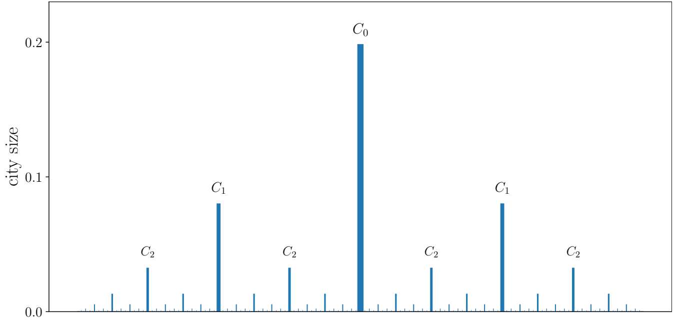


Figure 1: Illustration of optimal city hierarchy.

The size distribution follows a power law:

$$\ln(Rank) = -\frac{\ln(1/2)}{\ln(\theta)} \ln(Size) + C$$

Knowledge and Communication

- Hierarchical organization of knowledge (Garicano, 2000)
- A firm requires employees solve a set of problems $[0, 1]$
- A market for knowledge among management layers
- The ith layer is assigned m_i, learns to solve z_i at cost $c(z_i)$ and passes on the remainder $m_{i+1} = m_i - z_i$ to layer $i + 1$
- Price function p, communication cost τ

Knowledge and Communication

- The profits of the ith layer

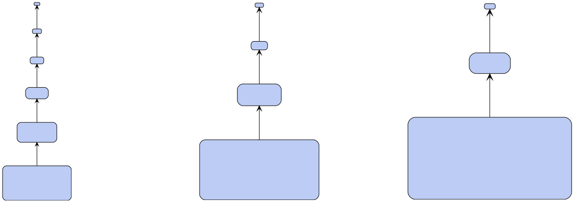
$$\pi_i = p(m_i) - c(z_i) - (1+\tau)p(m_i - z_i)$$

- The equilibrium price satisfies

$$p(m_i) = \min_{0 \le m_{i+1} \le m_i} \{c(m_i - m_{i+1}) + (1+\tau)p(m_{i+1})\}$$

- Assume that n employees can solve $f(n)$. Then $c(z) = wf^{-1}(z)$.
- $\{z_i\}$ is decreasing, so $\{n_i\}$ is decreasing

Knowledge and Communication



(a) $\tau = 0.2$ (b) $\tau = 0.4$ (c) $\tau = 0.6$

Figure 2: Optimal organizational structures.

Outline

Problem Description

- An agent takes actions $a(\cdot)$

$$v^*(\hat{x}) = \min_{a(t)} \int_0^\infty e^{\rho t} \ell(a(t))dt$$
$$\text{s.t.} \int_0^\infty a(t)dt = \hat{x}$$
$$a(t) \geq 0$$

- Assume that $\rho > 0$, $\ell(0) = 0$, $\ell' > 0$, and $\ell'' > 0$
- Define $x(t) = \hat{x} - \int_0^t a(s)ds = \int_t^\infty a(s)ds$

Problem Description

Theorem

1. The unique optimal solution is

$$a^*(t) = \arg\min_{a \geq 0} \left\{ e^{\rho t} \ell(a) + \lambda a \right\}$$

   where λ is a constant uniquely determined by $\int_0^\infty a^*(t) dt = \bar{x}$.

2. $v^*(x)$ is differentiable when $x > 0$ and satisfies

$$-\rho v^*(x) = \inf_{a \geq 0} \left\{ \ell(a) - (v^*)'(x) a \right\}$$

   with boundary condition $v^*(0) = 0$

3. The optimal action a^* at state x satisfies

$$-\rho v^*(x) = \inf_{a \geq 0} \left\{ \ell(a) - (v^*)'(x) a \right\} = \ell(a^*) - (v^*)'(x) a^*$$

Problem Description

Theorem (cont.)

4. The optimal action $a^*(t)$ is decreasing in t

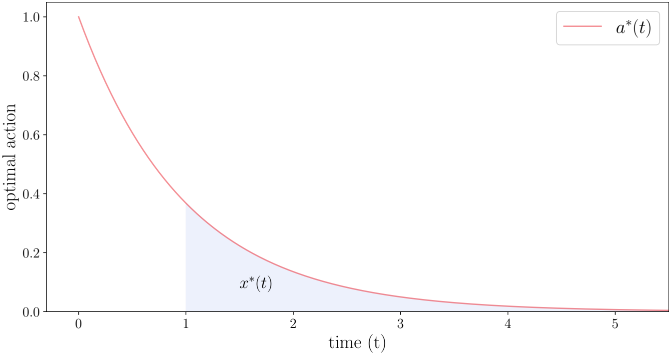


Figure 3: Optimal action and optimal state path

References

Garicano, L. (2000): "Hierarchies and the organization of knowledge in production," *Journal of Political Economy*, 108, 874–904.

Hsu, W.-T. (2012): "Central place theory and city size distribution," *The Economic Journal*, 122, 903–932.

Hsu, W.-T., T. J. Holmes, and F. Morgan (2014): "Optimal city hierarchy: A dynamic programming approach to central place theory," *Journal of Economic Theory*, 154, 245–273.

Kikuchi, T., K. Nishimura, and J. Stachurski (2018): "Span of control, transaction costs, and the structure of production chains," *Theoretical Economics*, 13, 729–760.