Coase Meets Bellman: Dynamic Programming for Production Networks

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Outline

Introduction

General Theory

Negative Discount DP

Applications

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Problem Description

\[ v^*(\hat{x}) := \min_{\{a_t\}} \sum_{t=0}^{\infty} \beta^t l(a_t) \]

s.t. \[ \sum_{t=0}^{\infty} a_t = \hat{x}, \ a_t \geq 0 \]

- The agent takes action \( a_t \) in period \( t \) and suffers loss \( l(a_t) \)
- Assume that \( \beta > 1, l(0) = 0, l' > 0, \) and \( l'' > 0 \)
- It doesn’t help to transform it to a maximization problem
  \[ \max_{\{a_t\}} \sum_{t=0}^{\infty} \beta^t (-l(a_t)) \]
- Lagrange multipliers?
Problem Description

\[ v^*(\hat{x}) := \min_{\{a_t\}} \sum_{t=0}^{\infty} \beta^t \ell(a_t) \]

s.t. \[ \sum_{t=0}^{\infty} a_t = \hat{x}, \ a_t \geq 0 \]

▶ Set \( x_0 = \hat{x} \) and \( x_{t+1} = x_t - a_t \)

▶ Does \( v^* \) satisfy the Bellman equation?

\[ v(x) = \min_{0 \leq a \leq x} \{ \ell(a) + \beta v(x - a) \} \]

▶ Why study the Bellman equation?

1. Use results from dynamic programming
2. Different economic interpretations
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An Abstract Dynamic Program

- A dynamic program consists of
  1. State space $X$
  2. Action space $A$
  3. Feasible correspondence $\Gamma(x) \subset A$
  4. Continuation aggregator $H(x, a, v)$

- The Bellman equation is
  \[ v(x) = (Tv)(x) := \inf_{a \in \Gamma(x)} H(x, a, v) \]

- Example: $X = A = [0, \hat{x}]$, $\Gamma(x) = [0, x]$, and
  \[ H(x, a, v) = \ell(a) + \beta v(x - a) \]
Fixed Point Results

Assumptions:

1. $H(x, a, v)$ is increasing and concave in $v$
2. There exist $\phi, \psi$ such that $T$ is a self-map on $[\phi, \psi]$
3. Other technical conditions

Theorem

$T$ has a unique fixed point $\bar{v} \in [\phi, \psi]$, $T^n v \to \bar{v}$ for all $v \in [\phi, \psi]$, and $\bar{\sigma}(x) := \arg \min_{a \in \Gamma(x)} H(x, a, \bar{v})$ is upper hemicontinuous. Under additional assumptions, $\bar{v}$ is increasing, convex, and continuously differentiable on $\text{int } X$ and $\bar{v}'(x) = H_x(x, \bar{\sigma}(x), \bar{v})$. 
Optimality

Definitions

- Any $\sigma$ corresponds to a policy operator
  \[
  (T_\sigma \nu)(x) := H(x, \sigma(x), \nu)
  \]
- For a sequence of feasible policies $\pi = (\sigma_0, \sigma_1, \ldots)$, define the $\pi$-value function
  \[
  \nu_\pi(x) = \lim_{n \to \infty} (T_{\sigma_0} T_{\sigma_1} \ldots T_{\sigma_n} \phi)(x)
  \]
- The value function is $\nu^*(x) := \inf_\pi \nu_\pi(x)$
- The optimal policy $\pi^*$ is such that $\nu^* = \nu_{\pi^*}$
- Stationary policies ($\sigma, \sigma, \ldots$)
Optimality

Assumptions:

1. If $v_n \uparrow v$, $H(x, a, v_n) \rightarrow H(x, a, v)$

2. There exists $\beta > 0$ such that for all $r > 0$

   $$H(x, a, v + r) \leq H(x, a, v) + \beta r$$

Theorem

The value function solves the Bellman equation and $\nu^* = \bar{\nu}$. There exists a stationary optimal policy $\sigma^*$ and $\sigma^* = \bar{\sigma}$. 
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Recall that $x_{t+1} = x_t - a_t$ and $x_0 = \hat{x}$. We aim to find $\sigma^*$ such that

$$a_t^* = \sigma^*(x_t) \text{ solves } v^*(\hat{x}) := \min_{\{a_t\}} \sum_{t=0}^{\infty} \beta^t \ell(a_t) \text{ s.t. } \sum_{t=0}^{\infty} a_t = \hat{x}, \ a_t \geq 0$$

Let $\phi(x) = \ell'(0)x$ and $\psi(x) = \ell(x)$

**Theorem**

1. $v^*$ is a fixed point of the Bellman operator

   $$ (Tv)(x) := \inf_{0 \leq a \leq x} \{\ell(a) + \beta v(x - a)\} $$

   and $T^n v \rightarrow v^*$ for all $v \in [\phi, \psi]$
Theorem (cont.)

2. There is a unique optimal policy given by

\[ \sigma^*(x) = \arg \min_{0 \leq a \leq x} \{ \ell(a) + \beta v^*(x - a) \}, \]

and \( \sigma^* \) is increasing.

3. \( v^* \) is strictly increasing, strictly convex, and continuously differentiable on \((0, \hat{x})\) with \((v^*)'(x) = \ell'(\sigma^*(x))\).

4. \( \{a_t^*\} \) is decreasing and satisfies

\[ \ell'(a_{t+1}^*) = \max \left\{ \frac{1}{\beta} \ell'(a_t^*), \ell'(0) \right\} \]
Negative Discount Dynamic Programming

Theorem

When $\ell'(0) = 0$, $\{a^*_t\}$ is optimal if and only if $\beta \ell'(a^*_{t+1}) = \ell'(a^*_t)$. The sequence is unique, decreasing, and strictly positive. Furthermore, $v^* = T v^*$.

To solve the Bellman equation:

- If $\ell'(0) > 0$, we use the Bellman operator directly
- If $\ell'(0) = 0$, we use the Euler equation
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Reinterpretation

- Time index → an index over decision making entities
- Tasks are completed sequentially

$$\text{total cost for agent 0} = \ell(a_0) + \beta \left[ \ell(a_1) + \beta \left( \ell(a_2) + \beta (\ldots) \right) \right]$$

- A social planner’s problem

$$v^*(\hat{x}) := \min_{\{a_t\}} \sum_{t=0}^{\infty} \beta^t \ell(a_t) \quad \text{s.t.} \quad \sum_{t=0}^{\infty} a_t = \hat{x}, \; a_t \geq 0$$

- A decentralized problem

$$v^*(x) := \min_{0 \leq a \leq x} \left\{ \ell(a) + \beta v^*(x - a) \right\}$$
Production Chains
Kikuchi et al. (2018)

► Competitive market with price-taking firms indexed by $i$

► Produce a unit of good: implementing a sequence of tasks

► The allocation of tasks $\{a_i\}$ satisfies $\sum_i a_i = 1$

► Define firm boundaries: $b_0 = 1$ and $b_{i+1} = b_i - a_i$

► Profits of the $i$th firm are

\[
\pi_i = p(b_i) - c(a_i) - (1 + \tau)p(b_{i+1})
\]

where $p$ is price, $c$ is production cost, and $\tau$ is transaction cost
Definition

The pair \((p, \{a_i\})\) is called an *equilibrium* for the production chain if

1. \(\pi_i = 0\) for all \(i\).

2. \(p(s) - c(s - t) - (1 + \tau)p(t) \leq 0\) for any pair \(s, t\) with \(0 \leq t \leq s \leq 1\), and

Theorem

Let \(\hat{x} = 1\), \(\ell = c\), and \(\beta = 1 + \tau\). Then, \((v^*, \{a_i^*\})\) is an equilibrium for the production chain.
City Hierarchy

- An alternative theory of city size distribution (Hsu, 2012; Hsu et al., 2014)
- Cities are built to host a continuum of dwellers of measure one
- A government opens competition for city developers
- Each developer builds a city of a certain size and pays other developers to build “satellite cities”
- Layers of cities are formed indexed by $i$
- Price function $p$, building cost $c$, tax rate $\tau$
City Hierarchy

- A developer assigned to host $s$ dwellers maximizes profits
  \[
  \max_{0 \leq t \leq s} \{ p(s) - c(s - t) - (1 + \tau)kp(t/k) \}
  \]

- The equilibrium price function satisfies
  \[
  p(s) = \min_{0 \leq t \leq s} \{ c(s - t) + (1 + \tau)kp(t/k) \}
  \]

- Let $c(s) = s^\gamma$ with $\gamma > 1$ and let $k = 2$
City Hierarchy

- The Euler equation: $c'(a^*_i) = (1 + \tau)c'(a^*_{i+1})$
- If $\theta := (1 + \tau)^{1/(1-\gamma)} < 1/2$, then $a^*_i = \theta^i(1 - 2\theta)$ and $v^*(s) = (1 - 2\theta)^{\gamma-1}s^\gamma$
- $p = v^*$ solves the Bellman equation and is an equilibrium price
City Hierarchy

The size distribution follows a power law:

$$\ln(Rank) = -\frac{\ln(1/2)}{\ln(\theta)} \ln(Size) + C$$

Figure 1: Illustration of optimal city hierarchy.
Knowledge and Communication

- Hierarchical organization of knowledge (Garicano, 2000)
- A firm requires employees solve a set of problems $[0, 1]$
- A market for knowledge among management layers
- The $i$th layer is assigned $m_i$, learns to solve $z_i$ at cost $c(z_i)$ and passes on the remainder $m_{i+1} = m_i - z_i$ to layer $i + 1$
- Price function $p$, communication cost $\tau$
Knowledge and Communication

- The profits of the $i$th layer
  \[ \pi_i = p(m_i) - c(z_i) - (1 + \tau)p(m_i - z_i) \]

- The equilibrium price satisfies
  \[ p(m_i) = \min_{0 \leq m_{i+1} \leq m_i} \left\{ c(m_i - m_{i+1}) + (1 + \tau)p(m_{i+1}) \right\} \]

- Assume that $n$ employees can solve $f(n)$. Then $c(z) = w f^{-1}(z)$.

- $\{z_i\}$ is decreasing, so $\{n_i\}$ is decreasing
Knowledge and Communication

Figure 2: Optimal organizational structures.
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Problem Description

- An agent takes actions $a(\cdot)$

$$v^*(\hat{x}) = \min_{a(t)} \int_0^\infty e^{\rho t} \ell(a(t)) \, dt$$

s.t. $\int_0^\infty a(t) \, dt = \hat{x}$

$a(t) \geq 0$

- Assume that $\rho > 0$, $\ell(0) = 0$, $\ell' > 0$, and $\ell'' > 0$

- Define $x(t) = \hat{x} - \int_0^t a(s) \, ds = \int_t^\infty a(s) \, ds$
### Theorem

1. The unique optimal solution is

   \[ a^*(t) = \arg \min_{a \geq 0} \{ e^{\rho t} \ell(a) + \lambda a \} \]

   where \( \lambda \) is a constant uniquely determined by \( \int_0^\infty a^*(t) dt = \bar{x} \).

2. \( v^*(x) \) is differentiable when \( x > 0 \) and satisfies

   \[-\rho v^*(x) = \inf_{a \geq 0} \{ \ell(a) - (v^*)'(x)a \} \]

   with boundary condition \( v^*(0) = 0 \).

3. The optimal action \( a^* \) at state \( x \) satisfies

   \[-\rho v^*(x) = \inf_{a \geq 0} \{ \ell(a) - (v^*)'(x)a \} = \ell(a^*) - (v^*)'(x)a^* \]
Theorem (cont.)

4. The optimal action $a^*(t)$ is decreasing in $t$

Figure 3: Optimal action and optimal state path
References


